

Paracontrolled distributions on Bravais lattices and weak universality of the 2d parabolic Anderson model

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Abstract

We develop a discrete version of paracontrolled distributions as a tool for deriving scaling limits of lattice systems, and we provide a formulation of paracontrolled distribution in weighted Besov spaces. Moreover, we develop a systematic martingale approach to control the moments of polynomials of i.i.d. random variables and to derive their scaling limits. As an application, we prove a weak universality result for the parabolic Anderson model: We study a nonlinear population model in a small random potential and show that under weak assumptions it scales to the linear parabolic Anderson model.

1 Introduction

Paracontrolled distributions were developed in [GIP15] to solve *singular SPDEs*, stochastic partial differential equations that are ill-posed because of the interplay of very irregular noise and nonlinearities. A typical example is the two-dimensional continuous parabolic Anderson model,

$$\partial_t u = \Delta u + u\xi - u\infty,$$

where $u: \mathbb{R}_+ \times \mathbb{R}^2 \rightarrow \mathbb{R}$ and ξ is a space white noise, the centered Gaussian distribution whose covariance is formally given by $\mathbb{E}[\xi(x)\xi(y)] = \delta(x - y)$. The irregularity of the white noise prevents the solution from being a smooth function, and therefore the product between u and the distribution ξ is not well defined. To make sense of it we need to eliminate some resonances between u and ξ by performing an infinite renormalization that replaces $u\xi$ by $u\xi - u\infty$. The motivation for studying singular SPDEs comes from mathematical physics, because they arise in the large scale description of natural microscopic dynamics. For example, if for the parabolic Anderson model we replace the white noise ξ by its periodization over a given box $[-L, L]^2$, then it was recently shown in [CGP17] that the solution u is the limit of $u^\varepsilon(t, x) = e^{-c^\varepsilon t} v^\varepsilon(t/\varepsilon^2, x/\varepsilon)$, where $v^\varepsilon: \mathbb{R}_+ \times \{-L/\varepsilon, \dots, L/\varepsilon\}^2 \rightarrow \mathbb{R}$ solves the lattice equation

$$\partial_t v^\varepsilon = \Delta^\varepsilon v^\varepsilon + \varepsilon v^\varepsilon \eta,$$

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where Δ^ε is the periodic discrete Laplacian and $(\eta(x))_{x \in \{-L/\varepsilon, \dots, L/\varepsilon\}^2}$ is an i.i.d. family of centered random variables with unit variance and sufficiently many moments.

Results of this type can be shown by relying more or less directly on paracontrolled distributions as they were developed in [GIP15] for functions of a continuous space parameter. But that approach comes at a cost because it requires us to control a certain random operator, which is highly technical and a difficulty that is not inherent to the studied problem. Moreover, it just applies to lattice models with polynomial nonlinearities. See the discussion below for details. Here we formulate a version of paracontrolled distributions that applies directly to functions on Bravais lattices and therefore provides a much simpler way to derive scaling limits and never requires us to bound random operators. Apart from simplifying the arguments, our new approach also allows us to study systems on infinite lattices that converge to equations on \mathbb{R}^d , while the formulation of the Fourier extension procedure we sketch below seems much more subtle in the case of an unbounded lattice. Moreover, we can now deal with non-polynomial nonlinearities which is crucial for our main application, a weak universality result for the parabolic Anderson model. Besides extending paracontrolled distributions to Bravais lattices we also develop paracontrolled distributions in weighted function spaces, which allows us to deal with paracontrolled equations on unbounded spaces that involve a spatially homogeneous noise. And finally we develop a general machinery for the use of discrete Wick contractions in the renormalization of discrete, singular SPDEs with i.i.d. noise which is completely analogous to the continuous Gaussian setting, and we build on the techniques of [CSZ17] to provide a criterion that identifies the scaling limits of discrete Wick products as multiple Wiener-Itô integrals.

Our main application is a weak universality result for the two-dimensional parabolic Anderson model. We consider a nonlinear population model $v^\varepsilon: \mathbb{R}_+ \times \mathbb{Z}^2 \rightarrow \mathbb{R}$,

$$\partial_t v^\varepsilon(t, x) = \Delta^{(d)} v^\varepsilon(t, x) + F(v^\varepsilon(t, x)) \eta^\varepsilon(x), \quad (1)$$

where $\Delta^{(d)}$ is the discrete Laplacian, $F \in C^2$ has a bounded second derivative and satisfies $F(0) = 0$, and $(\eta^\varepsilon(x))_{x \in \mathbb{Z}^2}$ is an i.i.d. family of random variables with $\text{Var}(\eta^\varepsilon(0)) = \varepsilon^2$ and $\mathbb{E}[\eta^\varepsilon(0)] = -F'(0)\varepsilon^2 c^\varepsilon$ for a suitable sequence of diverging constants $c^\varepsilon \sim |\log \varepsilon|$. The variable $v^\varepsilon(t, x)$ describes the population density at time t in the site x . The classical example would be $F(u) = u$, which corresponds to the discrete parabolic Anderson model in a small potential η^ε . In that case v^ε describes the evolution of a population where every individual performs an independent random walk and finds at every site x either favorable conditions if $\eta^\varepsilon(x) > 0$ that allow the individual to reproduce at rate $\eta^\varepsilon(x)$, or non-favorable conditions if $\eta^\varepsilon(x) < 0$ that kill the individual at rate $-\eta^\varepsilon(x)$. We can include some interaction between the individuals by choosing a nonlinear function F . For example, $F(u) = u(C - u)$ models a saturation effect which limits the overall population size in one site to C because of limited resources. In Section 5 we will prove the following result:

Theorem (see Theorem 5.10). *Assume that F and $(\eta^\varepsilon(x))$ satisfy the conditions described above and also that the p -th moment of $\eta^\varepsilon(0)$ is uniformly bounded in ε for some $p > 14$. Then there exists a unique solution v^ε to (1) with initial condition $v^\varepsilon(0, x) = \mathbf{1}_{x=0}$, up to a possibly finite explosion time T^ε with $T^\varepsilon \rightarrow \infty$ for $\varepsilon \rightarrow 0$, and $u^\varepsilon(t, x) = \varepsilon^{-2} v^\varepsilon(\varepsilon^{-2}t, \varepsilon^{-1}x)$ converges in law to the unique solution $u: \mathbb{R}_+ \times \mathbb{R}^2 \rightarrow \mathbb{R}$ of the linear continuous parabolic Anderson model*

$$\partial_t u = \Delta u + F'(0)u\xi - F'(0)^2 u\infty, \quad u(0) = \delta,$$

where δ denotes the Dirac delta.

Remark 1.1. *It may appear more natural to assume that $\eta^\varepsilon(0)$ is centered. However, we need the small shift of the expectation away from zero in order to create the renormalization $-F'(0)^2 u^\infty$ in the continuous equation. Making the mean of the variables $\eta^\varepsilon(x)$ slightly negative (assume $F|_{[0,\infty)} \geq 0$ so that $F'(0) \geq 0$) gives us a slightly higher chance for a site to be non-favorable than favorable. Without this, the population size would explode in the scale in which we look at it. A similar effect can also be observed in the Kac-Ising/Kac-Blume-Capel model, where the renormalization appears as a shift of the critical temperature away from its mean field value [MW17, SW16]. Note that in the linear case $F(u) = u$ we can always replace η^ε by $\eta^\varepsilon + c$ if we consider $e^{ct}v^\varepsilon(t)$ instead. So in that case it is not necessary to assume anything about the expectation of η^ε , we only have to adapt our reference frame to its mean.*

Structure of the paper Below we provide further references and explain in more details where to place our results in the current research in singular SPDEs and we fix some conventions and notations. In Sections 2- 4 we develop the theory of paracontrolled distributions on unbounded Bravais lattices, and in particular we derive Schauder estimates for quite general random walk semigroups. Section 5 contains the weak universality result for the parabolic Anderson model, and here we present our general methodology for dealing with multilinear functionals of independent random variables. The appendix contains several proofs that we outsourced. Finally, there is a list of important symbols at the end of the paper.

Related works As mentioned above, we can also use paracontrolled distributions for functions of a continuous space parameter to deal with lattice systems. The trick, which goes back at least to [MW17] and was inspired by [HM12], is to consider for a lattice function u^ε on say $\{k\varepsilon : -L/\varepsilon \leq k \leq L/\varepsilon\}^2$ the unique periodic function $\text{Ext}(u^\varepsilon)$ on $(\mathbb{R}/(2L\mathbb{Z}))^2$ whose Fourier transform is supported in $[-1/\varepsilon, 1/\varepsilon]^2$ and that agrees with u^ε in all the lattice points. If the equation for u^ε involves only polynomial nonlinearities, we can write down a closed equation for $\text{Ext}(u^\varepsilon)$ which looks similar to the equation for u^ε but involves a certain “Fourier shuffle” operator that is not continuous on the function spaces in which we would like to control $\text{Ext}(u^\varepsilon)$. But by introducing a suitable random operator that has to be controlled with stochastic arguments one can proceed to study the limiting behavior of $\text{Ext}(u^\varepsilon)$ and thus of u^ε . This argument has been applied to show the convergence of lattice systems to the KPZ equation [GP15b], the Φ_3^4 equation [ZZ15], and to the parabolic Anderson model [CGP17], and the most technical part of the proof was always the analysis of the random operator. The same argument was also applied to prove the convergence of the Kac-Ising / Kac-Blume-Capel model [MW17, SW16] to the Φ_2^4 / Φ_2^6 equation. This case can be handled without paracontrolled distributions, but also here some work is necessary to control the Fourier shuffle operator. This difficulty is of a technical nature and not inherent to the studied problems, and the line of argumentation we present here avoids that problem by analysing directly the lattice equation rather than trying to interpret it as a continuous equation.

Other intrinsic approaches to singular SPDEs on lattices have been developed in the context of regularity structures by Hairer and Matetski [HM15] and in the context of the semigroup approach to paracontrolled distributions by Bailleul and Bernicot [BB16], and we expect that both of these works could be combined with our martingale arguments of Section 5 to give an alternative proof of our weak universality result.

We call the convergence of the nonlinear population model to the linear parabolic Anderson model a “weak universality” result in analogy to the weak universality conjecture for

the KPZ equation. The (strong) KPZ universality conjecture states that a wide class of (1+1)-dimensional interface growth models scale to the same universal limit, the so called KPZ fixed point [MQR16], while the weak KPZ universality conjecture says that if we change some “asymmetry parameter” in the growth model to vanish at the right rate as we scale out, then the limit of this *family* of models is the KPZ equation. Similarly, here the influence of the random potential on the population model has taken as vanishing as we pass to the limit, so the parabolic Anderson model arises as scaling limit of a *family* of models. Similar weak universality results have recently been shown for other singular SPDEs such as the KPZ equation [GJ14, HQ15, GP15a, GP16] (this list is far from complete), the Φ_d^{2n} equations [MW17, HX16, SW16], or the (stochastic) nonlinear wave equation [GKO17, OT17].

Of course, a key task in singular stochastic PDEs is to renormalize and construct certain a priori ill-defined products between explicit stochastic processes. This already arises in rough paths [Lyo98] but there it is typically not necessary to perform any renormalizations and general construction and approximation results for Gaussian rough paths were developed in [FV10]. For singular SPDEs the constructions become much more involved and a general construction of regularity structures for equations driven by Gaussian noise was found only recently and is highly nontrivial [BHZ16, CH16]. For Gaussian noise it is natural to regroup polynomials of the noise in terms of Wick products, which goes back at least to [DD03] and is essentially always used in singular SPDEs, see [Hai13, Hai14, CC13, GP15b] and many more. Moreover, in the Gaussian case all moments of polynomials of the noise are equivalent, and therefore it suffices to control variances. In the non-Gaussian case we can still regroup in terms of Wick polynomials [MW17, HS15, CS16, SX16], but a priori the moments are no longer comparable and new methods are necessary. In [MW17] the authors used martingale inequalities to bound higher order moments in terms of variances.

In our case it may look as if there are no martingales around because the noise is constant in time. But if we enumerate the lattice points and sum up our i.i.d. variables along this enumeration, then we generate a martingale. This observation was used in [CGP17] to show that for certain polynomial functionals of the noise (“discrete multiple stochastic integrals”) the moments are still comparable, but the approach was somewhat ad-hoc and only applied directly to the product of two variables in “the first chaos”.

Here we develop here a general machinery for the use of discrete Wick contractions in the renormalization of discrete, singular SPDEs with i.i.d. noise which is completely analogous to the continuous Gaussian setting. Moreover, we build on the techniques of [CSZ17] to provide a criterion that identifies the scaling limits of discrete Wick products as multiple Wiener-Itô integrals. Although these techniques are only applied to the discrete $2d$ parabolic Anderson model, the approach extends in principle to any discrete formulation of popular singular SPDEs such as the KPZ equation or the Φ_d^4 models.

1.1 Conventions and Notation

We use the common notation \lesssim, \gtrsim in estimates to denote \leq, \geq up to a positive constant. The symbol \approx means that both \lesssim and \gtrsim hold true. For discrete indices we mean by $i \lesssim j$ that there is a $N \geq 0$ (independent of i, j) such that $i \leq j + N$ and similar for $j \gtrsim i$; the notation $i \sim j$ is shorthand for $i \lesssim j$ and $j \lesssim i$.

We denote partial derivatives by ∂^α for $\alpha \in \mathbb{N}^d$ and for $\alpha = (\mathbf{1}_{i=j})_j$ we write $\partial^i = \partial^\alpha$. The symbol ∂_v is reserved for the directional derivate in the direction of $v \in \mathbb{R}^d$. Our Fourier

transform follows the convention that for $f \in L^1(\mathbb{R}^d)$

$$\mathcal{F}_{\mathbb{R}^d} f(y) := \int_{\mathbb{R}^d} f(x) e^{-2\pi i x y} dx, \quad \mathcal{F}_{\mathbb{R}^d}^{-1} f(x) := \int_{\mathbb{R}^d} f(y) e^{2\pi i x y} dy.$$

The notations $\mathcal{F}, \mathcal{F}^{-1}, \hat{\cdot}, \cdot^\vee$ (without subscript “ \mathbb{R}^d ”) will be reserved for the Fourier transform on Bravais lattices which we introduce in Subsection 2.1. We denote by $*_{\mathbb{R}^d}$ the convolution on \mathbb{R}^d , the symbol $*$ is again reserved for the case of Bravais lattices, $f * g(x) = \sum_{z \in \mathcal{G}} |\mathcal{G}| f(x-z) g(z)$ with notation as in Subsection 2.1.

2 Weighted Besov spaces on Bravais lattices

2.1 Fourier transform on Bravais lattices

A *Bravais-lattice* in d dimensions consists of the integer combinations of d linearly independent vectors $a_1, \dots, a_d \in \mathbb{R}^d$, that is

$$\mathcal{G} := \mathbb{Z} a_1 + \dots + \mathbb{Z} a_d. \quad (2)$$

Given a Bravais lattice we define the basis $\hat{a}_1, \dots, \hat{a}_d$ of the reciprocal lattice by the requirement

$$\hat{a}_i \cdot a_j = \delta_{ij}, \quad (3)$$

and we set $\mathcal{R} := \mathbb{Z} \hat{a}_1 + \dots + \mathbb{Z} \hat{a}_d$. However, we will mostly work with the (centered) parallelepiped which is spanned by the basis vectors $\hat{a}_1, \dots, \hat{a}_d$:

$$\hat{\mathcal{G}} := [0, 1) \hat{a}_1 + \dots + [0, 1) \hat{a}_d - \frac{1}{2}(\hat{a}_1 + \dots + \hat{a}_d) = [-1/2, 1/2) \hat{a}_1 + \dots + [-1/2, 1/2) \hat{a}_d.$$

We call $\hat{\mathcal{G}}$ the *bandwidth* or *Fourier-cell* of \mathcal{G} to indicate that the Fourier transform of a map on \mathcal{G} lives on $\hat{\mathcal{G}}$ (see below). We also identify $\mathbb{R}^d/\mathcal{R} \simeq \hat{\mathcal{G}}$ and turn $\hat{\mathcal{G}}$ into an additive group which is invariant under translations by elements in \mathcal{R} .

Example 2.1. If we choose the canonical basis vectors $a_1 = e_1, \dots, a_d = e_d$, we have simply

$$\mathcal{G} = \mathbb{Z}^d, \quad \mathcal{R} = \mathbb{Z}^d, \quad \hat{\mathcal{G}} = \mathbb{T}^d = [-1/2, 1/2)^d.$$

Compare also the left lattice in Figure 2.1.

In Figure 2.1 we sketch some Bravais lattices \mathcal{G} together with their Fourier cells $\hat{\mathcal{G}}$. Note that the dashed lines between the points of the lattice are at this point a purely artistic supplement. However, they will become meaningful later on: If we imagine a particle performing a random walk on the lattice \mathcal{G} , then the dashed lines could be interpreted as the jumps it is allowed to undertake. From this point of view the lines are drawn by the diffusion operators we introduce in Section 3.

Definition 2.2. Given a Bravais lattice \mathcal{G} as defined in (2) we write

$$\mathcal{G}^\varepsilon := \varepsilon \mathcal{G}$$

for the sequence of Bravais lattice we obtain by dyadic rescaling with $\varepsilon = 2^{-N}$, $N \geq 0$. Whenever we say a statement (or an estimate) holds for \mathcal{G}^ε we mean that it holds (uniformly) for all $\varepsilon = 2^{-N}$, $N \geq 0$.

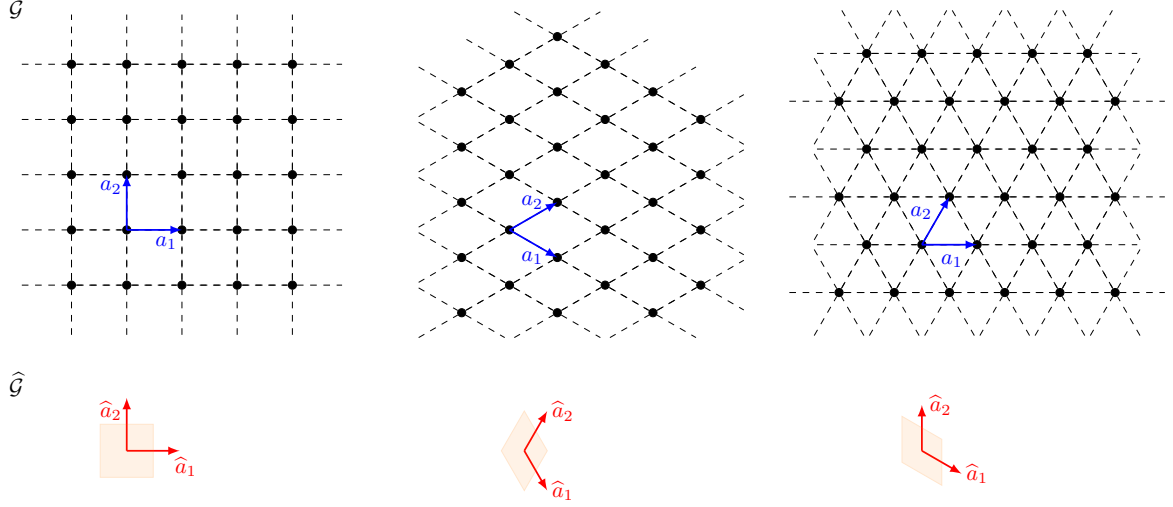


Figure 1: Sketch of some Bravais lattices \mathcal{G} with their bandwidths $\hat{\mathcal{G}}$: a square lattice, an oblique lattice and the so called hexagonal lattice. The length of the reciprocal vectors \hat{a}_i (and of $\hat{\mathcal{G}}$) is rather arbitrary since it actually depends on the units in which we measure a_i .

Remark 2.3. *The restriction to dyadic lattices fits well with the use of Littlewood-Paley theory which is traditionally build from dyadic decomposition. However, it turns out that we do not lose any generality by this. Indeed, all the statements and estimates below will hold uniformly as soon as we know that the scale of our lattice is contained in some interval $(c_1, c_2) \subset\subset (0, \infty)$. Therefore it is sufficient to group the members of any positive null-sequence $(\varepsilon_n)_{n \geq 0}$ in dyadic intervals $[2^{-(N+1)}, 2^{-N})$ to deduce the general statement.*

Given $\varphi \in \ell^1(\mathcal{G})$ we define its Fourier transform as

$$\mathcal{F}\varphi(x) := \hat{\varphi}(x) := |\mathcal{G}| \sum_{k \in \mathcal{G}} \varphi(k) e^{-2\pi i k \cdot x}, \quad x \in \hat{\mathcal{G}}, \quad (4)$$

where we introduced a “normalization constant” $|\mathcal{G}| := |\det(a_1, \dots, a_d)|$ that ensures that we obtain the usual Fourier transform on \mathbb{R}^d as $|\mathcal{G}|$ tends to 0. For the Fourier cell $\hat{\mathcal{G}}$ we will write $|\hat{\mathcal{G}}|$ for the Lebesgue measure of the cell.

If we consider $\mathcal{F}\varphi$ as a map on \mathbb{R}^d , then it is periodic under translations in \mathcal{R} . By the dominated convergence theorem $\mathcal{F}\varphi$ is continuous, so since $\hat{\mathcal{G}}$ is compact it is in $L^1(\hat{\mathcal{G}}) := L^1(\mathcal{G}, dx)$, where dx denotes integration with respect to the Lebesgue measure. For any $\psi \in L^1(\hat{\mathcal{G}})$ we define its inverse Fourier transform as

$$\mathcal{F}^{-1}\psi(k) := \check{\psi}(k) := \int_{\hat{\mathcal{G}}} \psi(x) e^{2\pi i k \cdot x} dx, \quad k \in \mathcal{G}. \quad (5)$$

Note that $|\mathcal{G}| = 1/|\hat{\mathcal{G}}|$ and therefore we get at least for φ with finite support $\mathcal{F}^{-1}\mathcal{F}\varphi = \varphi$. The Schwartz functions on \mathcal{G} are

$$\mathcal{S}(\mathcal{G}) := \left\{ \varphi: \mathcal{G} \rightarrow \mathbb{C} : \sup_{k \in \mathcal{G}} (1 + |k|)^m |\varphi(k)| < \infty \text{ for all } m \in \mathbb{N} \right\},$$

and we have $\mathcal{F}\varphi \in C^\infty(\widehat{\mathcal{G}})$ (with periodic boundary conditions) for all $\varphi \in \mathcal{S}(\mathcal{G})$, because for any multi-index $\alpha \in \mathbb{N}_0^d$ the dominated convergence theorem gives

$$\partial^\alpha \mathcal{F}\varphi(x) = |\mathcal{G}| \sum_{k \in \mathcal{G}} \varphi(k) (-2\pi i k)^\alpha e^{-2\pi i k \cdot x}.$$

By the same argument we have $\mathcal{F}^{-1}\psi \in \mathcal{S}(\mathcal{G})$ for all $\psi \in C^\infty(\widehat{\mathcal{G}})$, and as in the classical case $\mathcal{G} = \mathbb{Z}^d$ one can show that \mathcal{F} is an isomorphism from $\mathcal{S}(\mathcal{G})$ to $C^\infty(\widehat{\mathcal{G}})$ with inverse \mathcal{F}^{-1} . Many relations known from the \mathbb{Z}^d -case carry over readily to Bravais lattices such as Parseval's identity

$$\sum_{k \in \mathcal{G}} |\mathcal{G}| \cdot |\varphi(k)|^2 = \int_{|\widehat{\mathcal{G}}|} |\widehat{\varphi}(x)|^2 dx. \quad (6)$$

(to see this check for example with the Stone-Weierstrass theorem that $(|\mathcal{G}|^{1/2} e^{2\pi i k \cdot x})_{k \in \mathcal{G}}$ forms an orthonormal basis of $L^2(\widehat{\mathcal{G}}, dx)$) and the relation between convolution and multiplication

$$\mathcal{F}(\varphi_1 * \varphi_2)(x) = \mathcal{F}\left(\sum_{k \in \mathcal{G}} |\mathcal{G}| \varphi_1(k) \varphi_2(\cdot - k)\right)(x) = \mathcal{F}\varphi_1(x) \cdot \mathcal{F}\varphi_2(x), \quad (7)$$

$$\mathcal{F}^{-1}(\psi_2 *_{\widehat{\mathcal{G}}} \psi_1)(k) := \mathcal{F}^{-1}\left(\int_{\widehat{\mathcal{G}}} \psi_1(x) \psi_2(\cdot - x) dx\right)(k) = \mathcal{F}^{-1}\psi_1(k) \cdot \mathcal{F}^{-1}\psi_2(k). \quad (8)$$

Since $\mathcal{S}(\mathcal{G})$ consists of functions decaying faster than any polynomial, the Schwartz distributions on \mathcal{G} are the functions that grow at most polynomially,

$$\mathcal{S}'(\mathcal{G}) := \left\{ f : \mathcal{G} \rightarrow \mathbb{C} : \sup_{k \in \mathcal{G}} (1 + |k|)^{-m} |f(k)| < \infty \text{ for some } m \in \mathbb{N} \right\},$$

and $f(\varphi) := |\mathcal{G}| \sum_{k \in \mathcal{G}} f(k) \varphi(k)$ is well defined for $\varphi \in \mathcal{S}(\mathcal{G})$. We extend the Fourier transform to $\mathcal{S}'(\mathcal{G})$ by setting

$$(\mathcal{F}f)(\psi) := \widehat{f}(\psi) := f\left(\overline{\mathcal{F}^{-1}\psi}\right) = |\mathcal{G}| \sum_{k \in \mathcal{G}} f(k) \overline{\mathcal{F}^{-1}\psi}(k), \quad \psi \in C^\infty(\widehat{\mathcal{G}}),$$

where $\overline{\cdot}$ denotes the complex conjugate. This should be read as $(\mathcal{F}f)(\psi) = f(\mathcal{F}\psi)$, which however does not make any sense because for $\psi \in C^\infty(\widehat{\mathcal{G}})$ we did not define the Fourier transform $\mathcal{F}\psi$ but only $\mathcal{F}^{-1}\psi$. The Fourier transform $(\mathcal{F}f)(\psi)$ agrees with $\int_{\widehat{\mathcal{G}}} \widehat{f}(x) \psi(x) dx$ in case $f \in \mathcal{S}(\mathcal{G})$. It is possible to show that $\widehat{f} \in \mathcal{S}'(\widehat{\mathcal{G}})$, where

$$\mathcal{S}'(\widehat{\mathcal{G}}) := \{u : C^\infty(\widehat{\mathcal{G}}) \rightarrow \mathbb{C} : u \text{ is linear and } \exists C > 0, m \in \mathbb{N}_0 \text{ s.t. } |u(\psi)| \leq C \|\psi\|_{C_b^m(\widehat{\mathcal{G}})}\}$$

for $\|\psi\|_{C_b^m(\widehat{\mathcal{G}})} := \sum_{|\alpha| \leq m} \|\partial^\alpha \psi\|_{L^\infty(\widehat{\mathcal{G}})}$, and that \mathcal{F} is an isomorphism from $\mathcal{S}'(\mathcal{G})$ to $\mathcal{S}'(\widehat{\mathcal{G}})$ with inverse

$$(\mathcal{F}^{-1}u)(\varphi) := (\check{u})(\varphi) := |\mathcal{G}| \sum_{k \in \mathcal{G}} u(e^{2\pi i k \cdot}) \varphi(k). \quad (9)$$

As in the classical case $\mathcal{G} = \mathbb{Z}$ it is easy to see that we can identify every $f \in \mathcal{S}'(\mathcal{G})$ with a distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ by setting

$$f = |\mathcal{G}| \sum_{k \in \mathcal{G}} f(k) \delta(\cdot - k),$$

and we can identify any element $g \in \mathcal{S}'(\widehat{\mathcal{G}})$ of the frequency space with an \mathcal{R} -periodic distribution in $g \in \mathcal{S}'(\mathbb{R}^d)$ by setting

$$g(\varphi) = g\left(\sum_{k \in \mathcal{R}} \varphi(\cdot - k)\right), \quad \varphi \in \mathcal{S}(\mathbb{R}^d).$$

Conversely, every \mathcal{R} -periodic distribution $g \in \mathcal{S}'(\mathbb{R}^d)$ can be seen as element of $\mathcal{S}'(\widehat{\mathcal{G}})$, e.g. by considering $g(\varphi) := g(\psi\varphi)$, $\varphi \in C^\infty(\widehat{\mathcal{G}})$ where $\psi \in C_c^\infty(\mathbb{R}^d)$ is chosen such that $\sum_{k \in \mathcal{R}} \psi(\cdot - k) = 1$. This identification does not depend on the choice of ψ as can be easily checked and it motivates our definition of the extension operator \mathcal{E} below in Lemma 2.6.

With these identifications in mind we can now interpret the concepts introduced above as a sub-theory of the well-known Fourier analysis of tempered distributions. Whenever we mix both concepts, e.g. if we write

$$\phi \cdot f \tag{10}$$

for $f \in \mathcal{S}'(\widehat{\mathcal{G}})$ and $\phi \in \mathcal{S}(\mathbb{R}^d)$ (or even $\phi \in C^\infty(\mathbb{R}^d)$ using non-tempered distributions) this should be read in the sense of this broader theory. The identification follows the rule of thumb: *If an interpretation makes sense it is allowed; if there is more than one interpretation, then they all give the same result.*

Next, we want to introduce Besov spaces on \mathcal{G} . Recall that one way of constructing Besov spaces on \mathbb{R}^d is by making use of a dyadic partition of unity.

Definition 2.4. A dyadic partition of unity is a family $(\varphi_j)_{j \geq -1} \subset C_c^\infty(\mathbb{R}^d)$ of nonnegative radial functions such that

- $\text{supp } \varphi_{-1}$ is contained in a Ball around 0, $\text{supp } \varphi_j$ is contained in an annulus around 0 for $j \geq 0$,
- $\varphi_j = \varphi_0(2^{-j}\cdot)$ for $j \geq 0$,
- $\sum_{j \geq -1} \varphi_j(x) = 1$ for any $x \in \mathbb{R}^d$,
- If $|j - j'| > 1$ we have $\text{supp } \varphi_j \cap \text{supp } \varphi_{j'} = \emptyset$,

Using such a dyadic partition as a family of Fourier multipliers leads to the Littlewood-Paley blocks of a distribution $f \in \mathcal{S}'(\mathbb{R}^d)$,

$$\Delta_j f := \mathcal{F}_{\mathbb{R}^d}^{-1}(\varphi_j \mathcal{F}_{\mathbb{R}^d} f).$$

Each of these blocks is a smooth function and represents a “spectral chunk” of the distribution. By choice of the $(\varphi_j)_{j \geq -1}$ we have $f = \sum_{j \geq -1} \Delta_j f$ in $\mathcal{S}'(\mathbb{R}^d)$, and measuring the explosion/decay of the Littlewood-Paley blocks gives rise to the Besov spaces

$$\mathcal{B}_{p,q}^\alpha(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) : \|(2^{j\alpha} \|\Delta_j f\|_{L^p})_j\|_{\ell^q}\}. \tag{11}$$

In our case all the information about $f \in \mathcal{S}'(\mathcal{G})$ is stored in a finite bandwidth $\widehat{\mathcal{G}}$ and the Fourier transform \widehat{f} is periodic under translations in \mathcal{R} . Therefore, it is more natural to decompose only the compact set $\widehat{\mathcal{G}}$, and we could simply consider finitely many blocks. However, there is a small but delicate problem: We should decompose $\widehat{\mathcal{G}}$ in a smooth periodic

way, but if j is such that the support of φ_j touches the boundary of $\widehat{\mathcal{G}}$, the function φ_j will not necessarily be smooth in a periodic sense. We therefore redefine the dyadic partition of unity as

$$\varphi_j^{\mathcal{G}}(x) = \begin{cases} \varphi_j([x]), & j < j_{\mathcal{G}}, \\ 1 - \sum_{j < j_{\mathcal{G}}} \varphi_j([x]), & j = j_{\mathcal{G}}, \end{cases} \quad (12)$$

where $j \leq j_{\mathcal{G}} := \inf\{j : \text{supp } \varphi_j \cap \partial\widehat{\mathcal{G}} \neq \emptyset\}$ and $[x]$ is the (unique) $[x] \in \widehat{\mathcal{G}}$ such that $[x] - x \in \mathbb{Z}\widehat{a}_1 + \dots + \mathbb{Z}\widehat{a}_d$. Now we set

$$\Delta_j f = \Delta_j^{\mathcal{G}} f := \mathcal{F}^{-1}(\varphi_j^{\mathcal{G}} \mathcal{F} f).$$

We will often drop the index \mathcal{G} (on Δ_j and φ_j) when there is no risk of confusion with the Littlewood-Paley blocks for non-discrete distributions. As in the continuous case we will also use the notation $S_j^{\mathcal{G}} f = S_j f = \sum_{i < j} \Delta_i^{\mathcal{G}} f$.

Of course, for a fixed \mathcal{G} it may happen that $\Delta_{-1}^{\mathcal{G}} = \text{Id}$, but if we rescale the lattice \mathcal{G} to $\varepsilon\mathcal{G}$, the Fourier cell $\widehat{\mathcal{G}}$ changes to $\varepsilon^{-1}\widehat{\mathcal{G}}$ and so for $\varepsilon \rightarrow 0$ the following definition becomes meaningful.

Definition 2.5. *Given $\alpha \in \mathbb{R}$ and $p, q \in [1, \infty]$ we define*

$$\mathcal{B}_{p,q}^{\alpha}(\mathcal{G}) := \{f \in \mathcal{S}'(\mathcal{G}) \mid \|f\|_{\mathcal{B}_{p,q}^{\alpha}(\mathcal{G})} = \|(2^{j\alpha} \|\Delta_j^{\mathcal{G}} f\|_{L^p(\mathcal{G})})_j\|_{\ell^q} < \infty\},$$

where we define the $L^p(\mathcal{G})$ norm by

$$\|f\|_{L^p(\mathcal{G})} := \left(|\mathcal{G}| \sum_{k \in \mathcal{G}} |f(k)|^p \right)^{1/p} = \|\mathcal{G}|^{1/p} f\|_{\ell^p}. \quad (13)$$

We write furthermore $\mathcal{C}_p^{\alpha}(\mathcal{G}) := \mathcal{B}_{p,\infty}^{\alpha}(\mathcal{G})$.

The reader may have noticed that since we only consider finitely many $j = -1, \dots, j_{\mathcal{G}}$, the two spaces $\mathcal{B}_{p,q}^{\alpha}(\mathcal{G})$ and $L^p(\mathcal{G})$ are in fact identical with equivalent norms! However, since we are interested in uniform bounds on $\widehat{\mathcal{G}}^{\varepsilon}$ for $\varepsilon \rightarrow 0$, we are of course not allowed to switch between these spaces.

With the above constructions at hand it is easy to develop a theory of paracontrolled distributions on \mathcal{G} which is completely analogous to the one on \mathbb{R}^d . To prove the convergence of rescaled lattice models to models on the Euclidean space \mathbb{R}^d we need to compare discrete and continuous distributions, so we should extend the lattice model to a distribution in $\mathcal{S}'(\mathbb{R}^d)$. One way of doing so is to simply consider the identification with a Dirac comb, already mentioned above: $|\mathcal{G}| \sum_{k \in \mathcal{G}} f(k) \delta(\cdot - k) \in \mathcal{S}'(\mathbb{R}^d)$, but this has the disadvantage that the extension can only be controlled in spaces of quite low regularity because the Dirac delta has low regularity. We find the following extension convenient:

Lemma 2.6. *Let $\psi \in C_c^{\infty}(\mathbb{R}^d)$ be a positive function with $\sum_{k \in \mathcal{R}} \psi(\cdot - k) \equiv 1$ and set*

$$\mathcal{E}f := \mathcal{F}_{\mathbb{R}^d}^{-1}(\psi \cdot \mathcal{F}f), \quad f \in \mathcal{S}'(\mathcal{G}),$$

where the product $\psi \cdot \mathcal{F}f$ should be read as in (10). Then $\mathcal{E}f \in C^{\infty}(\mathbb{R}^d) \cap \mathcal{S}'(\mathbb{R}^d)$ and $\mathcal{E}f(k) = f(k)$ for all $k \in \mathcal{G}$.

Proof. We have $\mathcal{E}f \in \mathcal{S}'(\mathbb{R}^d)$ because (the periodic extension of) $\mathcal{F}f$ is in $\mathcal{S}'(\mathbb{R}^d)$, and therefore also $\mathcal{E}f = \mathcal{F}_{\mathbb{R}^d}^{-1}(\psi \mathcal{F}f) \in \mathcal{S}'(\mathbb{R}^d)$. Knowing that $\mathcal{E}f$ is in $\mathcal{S}'(\mathbb{R}^d)$, it must be in $C^\infty(\mathbb{R}^d)$ as well because it has compact spectral support by definition. Moreover, we can write for $k \in \mathcal{G}$

$$\mathcal{E}(f)(k) = \mathcal{F}f(\psi e^{-2\pi i k \cdot}) = \mathcal{F}f\left(\sum_{r \in \mathcal{R}} \psi(\cdot - r) e^{-2\pi i k(\cdot - r)}\right) = \mathcal{F}f(e^{-2\pi i k \cdot}) = f(k),$$

where in the first step $\mathcal{F}f$ should be again read as periodic distribution on \mathbb{R}^d as in (10) and where we used that $k \cdot \ell \in \mathbb{Z}$ for all $k \in \mathcal{G}$ and $\ell \in \mathcal{R}$. \square

It is possible to show that if \mathcal{E}^ε denotes the extension operator on \mathcal{G}^ε , then the family $(\mathcal{E}^\varepsilon)_{\varepsilon > 0}$ is uniformly bounded in $L(\mathcal{B}_{p,q}^\alpha(\mathcal{G}^\varepsilon), \mathcal{B}_{p,q}^\alpha(\mathbb{R}^d))$, and this can be used to obtain uniform regularity bounds for the extensions of a given family of lattice models.

However, since we are interested in equations with spatially homogeneous noise, we cannot expect the solution to be in $\mathcal{B}_{p,q}^\alpha(\mathcal{G})$ for any α, p, q and instead we have to consider weighted spaces. And in the case of the parabolic Anderson model it turns out to be convenient to even allow for subexponential growth of the form $e^{|\cdot|^\sigma}$ for $\sigma \in (0, 1)$, which means that we have to work on a larger space than $\mathcal{S}(\mathcal{G})$, where only polynomial growth is allowed. So before we proceed let us first recall the basics of the so called *ultra-distributions* on \mathbb{R}^d .

2.2 Ultra-distributions on Euclidean space

A drawback of Schwartz's theory of tempered distributions is the restriction that they can at most grow polynomially. As we will see later, it is convenient to allow our solution to have subexponential growth of the form $e^{\lambda|\cdot|^\sigma}$ for $\sigma \in (0, 1)$ and $\lambda > 0$. It is therefore necessary to work in a larger space $\mathcal{S}'_\omega(\mathbb{R}^d) \supseteq \mathcal{S}'(\mathbb{R}^d)$, the space of so called (*tempered*) *ultra-distributions*, which has less restrictive growth conditions but on which one still has a Fourier transform. Similar techniques already appear in the context of singular SPDEs in [MW15], where the authors use Gevrey functions that are characterized by a condition similar to the one in Definition 2.8 below. Here we will follow a slightly different approach that goes back to Beurling and Björck [Bjö66], and which mimics essentially the definition of tempered distribution via Schwartz functions. For a broader introduction to ultra-distributions see for example [Tri83, Chapter 6] or [Bjö66].

Throughout this paper, ω will be one of the following radial functions on \mathbb{R}^d

$$\omega(x) = \log(1 + |x|) \tag{14}$$

or

$$\omega(x) = |x|^\sigma, \sigma \in (0, 1). \tag{15}$$

Tempered ultra-distributions are essentially those distributions that grow at most like a power of e^ω . The classical Fourier theory of tempered distributions is governed by the triple

$$\mathcal{D}(\mathbb{R}^d) \subseteq \mathcal{S}(\mathbb{R}^d) \subseteq C^\infty(\mathbb{R}^d),$$

where we write $\mathcal{D}(\mathbb{R}^d) = C_c^\infty(\mathbb{R}^d)$ for the space of test functions. In the theory of ultra-distributions these spaces are replaced by

$$\mathcal{D}_\omega(\mathbb{R}^d) \subseteq \mathcal{S}_\omega(\mathbb{R}^d) \subseteq C_\omega^\infty(\mathbb{R}^d),$$

which have more restrictive growth conditions as specified in the following two definitions.

Definition 2.7. Let ω be as in (14) or (15). For $f \in \mathcal{S}(\mathbb{R}^d)$, $\lambda > 0$, and $\alpha \in \mathbb{N}^d$ we define

$$p_{\alpha,\lambda}(f) = \sup_{x \in \mathbb{R}^d} e^{\lambda\omega(x)} |\partial^\alpha f(x)|, \quad (16)$$

$$\pi_{\alpha,\lambda}(f) = \sup_{x \in \mathbb{R}^d} e^{\lambda\omega(x)} |\partial^\alpha \mathcal{F}_{\mathbb{R}^d} f(x)|. \quad (17)$$

We define a locally convex space $\mathcal{S}_\omega(\mathbb{R}^d)$ by

$$\mathcal{S}_\omega(\mathbb{R}^d) := \left\{ f \in \mathcal{S}(\mathbb{R}^d) \mid p_{\alpha,\lambda}(f) < \infty, \pi_{\alpha,\lambda}(f) < \infty \text{ for any } \lambda > 0, \alpha \in \mathbb{N}^d \right\},$$

equipped with the semi-norms (16) and (17). Its topological dual $\mathcal{S}'_\omega(\mathbb{R}^d) := (\mathcal{S}_\omega(\mathbb{R}^d))'$ is equipped with the strong topology. We will also use the ultra-differentiable test functions

$$\mathcal{D}_\omega(\mathbb{R}^d) = \mathcal{S}_\omega(\mathbb{R}^d) \cap C_c^\infty(\mathbb{R}^d).$$

Given a compact set $K \subseteq \mathbb{R}^d$ we write $\mathcal{D}_\omega(K)$ for the set of $f \in \mathcal{D}_\omega(\mathbb{R}^d)$ with $\text{supp } f \subseteq K$.

If ω is of the form (14) we have $\mathcal{S}_\omega = \mathcal{S}$ (with the same topology) and $\mathcal{D}_\omega(\mathbb{R}^d) = C_c^\infty(\mathbb{R}^d)$. In the case (15) \mathcal{S}_ω is strictly larger than \mathcal{S} , indeed $e^{c|\cdot|^{\sigma'}} \in \mathcal{S}'_\omega \setminus \mathcal{S}'$ for $\sigma' \in (0, \sigma)$. If ω is of the form (15) the case $\sigma = 1$ must not be included, since it would imply that the Fourier transform of any $f \in \mathcal{D}_\omega(\mathbb{R}^d)$ is bounded by $e^{-c|x|}$, $c > 0$, which means that f is analytic and of compact support and thus $f = 0$. The case $\sigma = 1$ does therefore not allow for localization and in particular there is no hope of having a Littlewood-Paley theory.

The role of the smooth functions $C^\infty(\mathbb{R}^d)$ is played by the so called ultra-differentiable functions $C_\omega^\infty(\mathbb{R}^d)$.

Definition 2.8. For ω as in (15) and an open set $U \subseteq \mathbb{R}^d$ we say that $f \in C^\infty(U)$ is ultra-differentiable and write $f \in C_\omega^\infty(U)$ if we can find for every compact set $K \subseteq U$ and $\varepsilon > 0$ a constant $C_{\varepsilon,K} \geq 0$ such that

$$\sup_K |\partial^\alpha f| \leq C_{\varepsilon,K} \cdot \varepsilon^{|\alpha|} (\alpha!)^{1/\sigma}, \quad (18)$$

for all multi-indices $\alpha \in \mathbb{N}^d$. Taking the minimal choice of the constants $C_{\varepsilon,K}$ gives a family of semi-norms that equips $C_\omega^\infty(U)$ with a topology. If ω is of the form (14) we set $C_\omega^\infty(U) = C^\infty(U)$.

Remark 2.9. The factor $\alpha!$ in (18) can also be replaced by $|\alpha|!$ or $|\alpha|^{|\alpha|}$ [Rod93, Proposition 1.4.2] as can be easily seen from $\alpha! \leq |\alpha|! \leq d^{|\alpha|} \alpha!$ and Stirling's formula.

The following lemma clarifies the relation of $C_\omega^\infty(\mathbb{R}^d)$ with the spaces of Definition 2.7 and gives a sometimes more practical characterization of $\mathcal{D}_\omega(\mathbb{R}^d)$.

Lemma 2.10. The space $C_\omega^\infty(\mathbb{R}^d)$ is stable under addition, multiplication, composition and, if well defined, division. Furthermore, $\mathcal{D}_\omega(\mathbb{R}^d)$ is simply the space of compactly supported functions in $C_\omega^\infty(\mathbb{R}^d)$:

$$\mathcal{D}_\omega(\mathbb{R}^d) = C_\omega^\infty(\mathbb{R}^d) \cap \mathcal{D}(\mathbb{R}^d). \quad (19)$$

The space $\mathcal{S}_\omega(\mathbb{R}^d)$ is stable under addition, multiplication and convolution, and we have

$$\mathcal{S}_\omega(\mathbb{R}^d) \subseteq C_\omega^\infty(\mathbb{R}^d).$$

Proof. The proof is sketched in the appendix. \square

The dual of $\mathcal{D}_\omega(\mathbb{R}^d)$ is the space of *ultra-distributions*

$$\mathcal{D}'_\omega(\mathbb{R}^d) \supseteq \mathcal{S}'_\omega(\mathbb{R}^d),$$

and many linear operations such as addition or derivation that can be defined on distributions can be translated immediately to this new space. We see with (19) that $C^\infty_\omega(\mathbb{R}^d)$ should be interpreted as the set of permitted smooth multipliers for ultra-distributions. The space of *tempered ultra-distributions* $\mathcal{S}'_\omega(\mathbb{R}^d)$ is small enough to allow for a Fourier transform.

Definition 2.11. For $f \in \mathcal{S}'_\omega(\mathbb{R}^d)$ and $\varphi \in \mathcal{S}_\omega(\mathbb{R}^d)$ we set

$$\begin{aligned} \mathcal{F}_{\mathbb{R}^d} f(\varphi) &:= \widehat{f}(\varphi) := f(\mathcal{F}_{\mathbb{R}^d} \varphi), \\ \mathcal{F}_{\mathbb{R}^d}^{-1} f(\varphi) &:= \check{f}(\varphi) := f(\mathcal{F}_{\mathbb{R}^d}^{-1} \varphi). \end{aligned}$$

By definition of $\mathcal{S}_\omega(\mathbb{R}^d)$ we have that $\mathcal{F}_{\mathbb{R}^d}$ and $\mathcal{F}_{\mathbb{R}^d}^{-1}$ are isomorphisms on $\mathcal{S}_\omega(\mathbb{R}^d)$ which implies that $\mathcal{F}_{\mathbb{R}^d}$ and $\mathcal{F}_{\mathbb{R}^d}^{-1}$ are isomorphisms on $\mathcal{S}'_\omega(\mathbb{R}^d)$.

The following lemma proves that the set of compactly supported ultra-differentiable functions $\mathcal{D}_\omega(\mathbb{R}^d)$ is rich enough to localize ultra-distributions, which gets the Littlewood-Paley theory started.

Lemma 2.12 ([Bjö66], Theorem 1.3.7.). Let ω be of the form (14) or (15). For every pair of compact sets $K \subsetneq K' \subseteq \mathbb{R}^d$ there is a $\varphi \in \mathcal{D}_\omega(\mathbb{R}^d)$ such that

$$\varphi|_K = 1, \quad \text{supp } \varphi \subseteq K'.$$

2.3 Ultra-distributions on Bravais lattices

For the discrete setup we essentially proceed as in Subsection 2.1 and define spaces

$$\mathcal{S}_\omega(\mathcal{G}) = \left\{ f : \mathcal{G} \rightarrow \mathbb{C} \mid \sup_{k \in \mathcal{G}} e^{\lambda \omega(k)} |f(k)| < \infty \text{ for all } \lambda > 0 \right\},$$

and their duals (when equipped with the natural topology)

$$\mathcal{S}'_\omega(\mathcal{G}) = \left\{ f : \mathcal{G} \rightarrow \mathbb{C} \mid \sup_{k \in \mathcal{G}} e^{-\lambda \omega(k)} |f(k)| < \infty \text{ for some } \lambda > 0 \right\},$$

with the pairing $f(\varphi) = |\mathcal{G}| \sum_{k \in \mathcal{G}} f(k) \varphi(k)$, $\varphi \in \mathcal{S}_\omega(\mathcal{G})$. As in Subsection 2.1 we can then define a Fourier transform on $\mathcal{S}'_\omega(\mathcal{G})$ which maps the discrete space $\mathcal{S}_\omega(\mathcal{G})$ into the space of ultra-differentiable functions $\mathcal{S}_\omega(\widehat{\mathcal{G}}) := C^\infty_\omega(\widehat{\mathcal{G}})$ with periodic boundary conditions. The dual space $\mathcal{S}'_\omega(\widehat{\mathcal{G}})$ can then be equipped with a Fourier transform \mathcal{F}^{-1} as in (9) such that $\mathcal{F}, \mathcal{F}^{-1}$ become isomorphisms between $\mathcal{S}'_\omega(\mathcal{G})$ and $\mathcal{S}'_\omega(\widehat{\mathcal{G}})$ that are inverse to each other. For a proof of these statements we refer to Lemma A.1.

Performing identifications as in the case of $\mathcal{S}'(\mathbb{R}^d)$ we can see these concepts as a sub-theory of the Fourier analysis on $\mathcal{S}'_\omega(\mathbb{R}^d)$ with the only difference that we have to choose the function ψ with $\sum_{k \in \mathcal{R}} \psi(\cdot - k) = 1$ on page 8 as an element of $\mathcal{D}_\omega(\mathbb{R}^d)$.

2.4 Discrete weighted Besov spaces

We now introduce discrete, weighted Besov spaces. As weights we allow for functions ρ whose growth can be controlled by ω .

Definition 2.13 ([Tri83], Definition 6.2.1). *Given ω as in (14) or (15) we define $\boldsymbol{\rho}(\omega)$ as the set of measurable, strictly positive functions $\rho: \mathbb{R}^d \rightarrow (0, \infty)$ such that for some $\lambda > 0$, uniformly in $x, y \in \mathbb{R}^d$,*

$$\rho(x) \lesssim \rho(y) e^{\lambda \omega(x-y)}. \quad (20)$$

Note that $\boldsymbol{\rho}(\omega)$ is stable under addition and multiplication.

The bound (20) is necessary to control convolutions in weighted norms, as we will explain in more detail below. The only weights we will explicitly use in this paper are polynomial weights

$$p^\kappa(x) = \frac{1}{(1 + |x|)^\kappa} \in \boldsymbol{\rho}(\log(1 + |\cdot|)) \subseteq \boldsymbol{\rho}(|\cdot|^\sigma)$$

for $\kappa > 0$, $\sigma \in (0, 1)$ and sub-exponential weights

$$e_{l+t}^\sigma(x) = e^{-(l+t)(1+|x|)^\sigma} \in \boldsymbol{\rho}(|\cdot|^\sigma)$$

for $\sigma \in (0, 1)$, $l \in \mathbb{R}$ and a parameter $t \geq 0$ which later we will identify with a time variable. This choice was inspired by [HL15], the only difference is that they consider $\sigma = 1$ which is not permitted for us as explained in Subsection 2.2.

We can now give our definition of a discrete, weighted Besov space, where we essentially proceed as in Subsection 2.1 with the only difference that $\rho \in \boldsymbol{\rho}(\omega)$ is included in the definition and that our partition of unity must now be chosen in $C^\infty(\mathbb{R}^d)$: We take a partition of unity $\varphi_j \in \mathcal{D}_\omega(\mathbb{R}^d)$ on \mathbb{R}^d (with Lemma 2.12) and then modify as in 2.1 the first function $\varphi_{j_\mathcal{G}}$ that touches with its support $\partial \hat{\mathcal{G}}$ as in (12). This gives a (periodically) smooth decomposition of $\hat{\mathcal{G}}$ with $\varphi_j \in \mathcal{D}_\omega(\mathbb{R}^d)$ for $j < j_\mathcal{G}$. We might again drop the index \mathcal{G} if there is no risk of confusion. If we consider a sequence of Bravais lattices \mathcal{G}^ε we choose a common partition of unity on \mathbb{R}^d which gives $\varphi_j^{\mathcal{G}^\varepsilon}$ that are independent of ε as long as $j < j_{\mathcal{G}^\varepsilon}$.

Definition 2.14. *Given $\alpha \in \mathbb{R}$, $p, q \in [1, \infty]$ and $\rho \in \boldsymbol{\rho}(\omega)$ for ω as in (14) or (15) we define*

$$\mathcal{B}_{p,q}^\alpha(\mathcal{G}, \rho) := \{f \in \mathcal{S}'_\omega(\mathcal{G}) \mid \|f\|_{\mathcal{B}_{p,q}^\alpha(\mathcal{G}, \rho)} := \|(2^{j\alpha} \|\rho \Delta_j^\mathcal{G} f\|_{L^p(\mathcal{G})})_j\|_{\ell^q} < \infty\},$$

where the dyadic partition of unity is constructed as explained above. We write furthermore $\mathcal{C}_p^\alpha(\mathcal{G}, \rho) = \mathcal{B}_{p,\infty}^\alpha(\mathcal{G}, \rho)$ and define

$$L^p(\mathcal{G}, \rho) := \{f \in \mathcal{S}_\omega(\mathcal{G}) \mid \|f\|_{L^p(\mathcal{G}, \rho)} := \|\rho f\|_{L^p(\mathcal{G})} < \infty\},$$

$$\text{i.e. } \|f\|_{\mathcal{B}_{p,q}^\alpha(\mathcal{G}, \rho)} = \|(2^{j\alpha} \|\Delta_j^\mathcal{G} f\|_{L^p(\mathcal{G}, \rho)})_j\|_{\ell^q}.$$

The translation of this definition to continuous spaces $\mathcal{B}_{p,q}^\alpha(\mathbb{R}^d, \rho)$, $\mathcal{C}_p^\alpha(\mathbb{R}^d, \rho)$ and $L^p(\mathbb{R}^d, \rho)$ is immediate.

Remark 2.15. When we introduce the weight we have a choice where to put it. Here we set $\|f\|_{L^p(\mathcal{G}, \rho)} = \|\rho f\|_{L^p(\mathcal{G})}$, which is analogous to [Tri83] or [HL15], but different from [MW15] who instead take the L^p norm under the measure $\rho(x)dx$. For $p = 1$ both definitions coincide, but for $p = \infty$ the weighted L^∞ space of Mourrat and Weber does not feel the weight at all and it coincides with its unweighted counterpart.

The Littlewood-Paley blocks that we used to construct the discrete Besov space in Definition 2.14 have the useful property that they can be written for a sequence \mathcal{G}^ε as

$$\Delta_j^{\mathcal{G}^\varepsilon} f = \varphi_j^{\mathcal{G}^\varepsilon}(D)f = K_j * f,$$

where $K_j = \mathcal{F}^{-1}\varphi_j^{\mathcal{G}^\varepsilon} = 2^{jd}\mathcal{K}(2^j\cdot)$ with a $\mathcal{K} \in \mathcal{S}_\omega(\mathbb{R}^d)$ that depends on whether $j = -1$, $j \in \{0, \dots, j_{\mathcal{G}^\varepsilon} - 1\}$, or $j = j_{\mathcal{G}^\varepsilon}$ and on \mathcal{G} , but not on ε . This is a consequence of the dyadic scaling of our lattice (see Lemma A.2) and will be helpful in translating arguments from the continuous theory into our discrete framework. We will suppress in our notation the dependency of K_j and \mathcal{K} on the three cases for j and on \mathcal{G} to uncluster the notation a bit. The convolution $*$ should be read in the sense of \mathcal{G}^ε , i.e.

$$K_j * f(x) = \sum_{k \in \mathcal{G}^\varepsilon} |\mathcal{G}^\varepsilon| K_j(x - k)f(k) = \sum_{k \in \mathcal{G}^\varepsilon} |\mathcal{G}^\varepsilon| 2^{jd}\mathcal{K}(2^j(x - k))f(k). \quad (21)$$

Let us stress the fact that \mathcal{K} is defined on all of \mathbb{R}^d , and therefore (21) actually makes sense for all $x \in \mathbb{R}^d$. For a suitable choice of $\mathcal{K} \in \mathcal{S}_\omega(\mathbb{R}^d)$ (precisely the one in Lemma A.2) this smooth extension coincides with the action of the extension operator \mathcal{E}^ε that we will introduce in Subsection 2.5 below.

A typical example for a computation in this paper would be the task to bound for a given $K \in \mathcal{S}_\omega(\mathbb{R}^d)$ an object like $\|\rho(2^i K(2^{id}\cdot) * g)\|_{L^p(\mathcal{G}^\varepsilon)}$ for $i \leq j_{\mathcal{G}^\varepsilon}$ and $\rho \in \boldsymbol{\rho}(\omega)$ by $\|\rho g\|_{L^p(\mathcal{G}^\varepsilon)}$, which follows with (20) from the Young inequality on \mathcal{G}^ε if we know that $\|2^i K(2^{id}\cdot)\|_{L^1(\mathcal{G}^\varepsilon, e^{\lambda\omega})} \lesssim 1$, see Lemma 2.16 below. This mechanism allows us to carry over many results from the continuous Littlewood-Paley theory to our discrete, weighted setting. For example we see immediately that $\|\Delta_j^{\mathcal{G}} g\|_{L^p(\mathcal{G}^\varepsilon, \rho)} = \|2^{jd}\mathcal{K}(2^j\cdot) * g\|_{L^p(\mathcal{G}^\varepsilon, \rho)} \lesssim \|g\|_{L^p(\mathcal{G}^\varepsilon, \rho)}$.

Interpreting the \mathcal{G}^ε convolution $2^i K(2^{id}\cdot) * g(x) = \sum_{k \in \mathcal{G}^\varepsilon} |\mathcal{G}^\varepsilon| 2^{jd}K(2^j(x - k))g(k)$ as a function on $x \in \mathbb{R}^d$ we can also bound it in the $L^p(\mathbb{R}^d, \rho)$ norm, compare Lemma 2.16.

Lemma 2.16. *Given \mathcal{G}^ε as in Definition 2.2 and $\Phi \in \mathcal{S}_\omega(\mathbb{R}^d)$ we have for any $j \geq -1$ with $2^j \lesssim \varepsilon^{-1}$ and $p \in [1, \infty]$, $\lambda > 0$ for $\Phi_j := 2^{jd}\Phi(2^j\cdot)$*

$$\|\Phi_j\|_{L^p(\mathcal{G}^\varepsilon, e^{\lambda\omega})} \lesssim 2^{jd(1-1/p)},$$

where the implicit constant is independent of $\varepsilon > 0$. We even have the stronger result

$$\sup_{x \in \mathbb{R}^d} \|\Phi_j(\cdot + x)\|_{L^p(\mathcal{G}^\varepsilon, e^{\lambda\omega(\cdot+x)})} \lesssim 2^{jd(1-1/p)}. \quad (22)$$

In particular we have for $\rho \in \boldsymbol{\rho}(\omega)$

$$\|\Phi_j * f\|_{L^p(\mathcal{G}^\varepsilon, \rho)} \lesssim \|f\|_{L^p(\mathcal{G}^\varepsilon, \rho)}, \quad \|\Phi_j * f\|_{L^p(\mathbb{R}^d, \rho)} \lesssim \|f\|_{L^p(\mathcal{G}^\varepsilon, \rho)},$$

where the extension of $\Phi_j * f$ to \mathbb{R}^d in the second estimate should be read as above.

Proof. Without loss of generality we assume $j \geq 0$. The case $p = \infty$ follows from the definition of $\mathcal{S}_\omega(\mathbb{R}^d)$ and $e^{\lambda\omega(k)} \leq e^{\lambda\omega(2^j k)}$, so that we only have to show the statement for $p < \infty$. And indeed we obtain

$$\begin{aligned} \|\Phi_j\|_{L^p(\mathcal{G}^\varepsilon, e^{\lambda\omega})}^p &= \sum_{k \in \mathcal{G}^\varepsilon} |\mathcal{G}^\varepsilon| |\Phi_j(k)|^p e^{p\lambda\omega(k)} = 2^{jdp} \varepsilon^d \sum_{k \in \mathcal{G}} |\mathcal{G}| |\Phi(2^j \varepsilon k)|^p e^{p\lambda\omega(\varepsilon k)} \\ &\leq 2^{jdp} \varepsilon^d \sum_{k \in \mathcal{G}} |\mathcal{G}| |\Phi(2^j \varepsilon k)|^p e^{p\lambda\omega(2^j \varepsilon k)} \lesssim 2^{jd(p-1)} \sum_{k \in \mathcal{G}} |\mathcal{G}| 2^{jd} \varepsilon^d \frac{1}{1 + |2^j \varepsilon k|^{d+1}} \\ &\stackrel{\text{Lemma A.3}}{\lesssim} 2^{jd(p-1)} \int_{\mathbb{R}^d} dz (2^j \varepsilon)^d \frac{1}{1 + |2^j \varepsilon z|^{d+1}} \lesssim 2^{jd(p-1)}, \end{aligned}$$

where we used that $\Phi \in \mathcal{S}_\omega(\mathbb{R}^d)$ and in the application of Lemma A.3 that for $|x - y| \lesssim 1$ the quotient $\frac{1 + |2^j \varepsilon x|}{1 + |2^j \varepsilon y|}$ is uniformly bounded. Inequality (22) can be proved in the same way since it suffices to take the supremum over $|x| \lesssim \varepsilon$.

The estimates on $\Phi_j * f$ then follow by Young's inequality on \mathcal{G}^ε and a mixed Young inequality, Lemma A.5, together with (20). \square

As in the continuous case we can state an embedding theorem for discrete Besov spaces. Since it can be shown exactly as its continuous cousin we will not give its proof here.

Lemma 2.17. *For any $\alpha_1 \in \mathbb{R}$, $1 \leq p_1 \leq p_2 \leq \infty$, $1 \leq q_1 \leq q_2 \leq \infty$ and weights ρ_1, ρ_2 with $\rho_2 \lesssim \rho_1$ we have the continuous embedding (with norm independent of $\varepsilon \in (0, 1]$)*

$$\mathcal{B}_{p_1, q_1}^{\alpha_1}(\mathcal{G}^\varepsilon, \rho_1) \subseteq \mathcal{B}_{p_2, q_2}^{\alpha_2}(\mathcal{G}^\varepsilon, \rho_2)$$

for $\alpha_2 \leq \alpha_1 - d(1/p_1 - 1/p_2)$.

For later purposes we also recall the continuous version of this embedding.

Lemma 2.18 ([ET96], Theorem 4.2.3). *For any $\alpha_1 \in \mathbb{R}$, $1 \leq p_1 \leq p_2 \leq \infty$, $1 \leq q_1 \leq q_2 \leq \infty$ and weights ρ_1, ρ_2 with $\rho_2 \lesssim \rho_1$ we have the continuous embedding (with norm independent of $\varepsilon \in (0, 1]$)*

$$\mathcal{B}_{p_1, q_1}^{\alpha_1}(\mathbb{R}, \rho_1) \subseteq \mathcal{B}_{p_2, q_2}^{\alpha_2}(\mathbb{R}, \rho_2)$$

for $\alpha_2 \leq \alpha_1 - d(1/p_1 - 1/p_2)$. If $\alpha_2 < \alpha_1 - d(1/p_1 - 1/p_2)$ and $\lim_{|x| \rightarrow \infty} \rho_2(x)/\rho_1(x) = 0$ the embedding is compact.

2.5 The extension operator

From now on, we fix a partition of unity (φ_j) in $\mathcal{D}_\omega(\mathbb{R}^d)$ that then gives a non-trivial partition $\varphi_j^\mathcal{G}$ of the bandwidth $\widehat{\mathcal{G}}$, i.e. $j_\mathcal{G} \neq -1$. We choose a symmetric function $\psi \in \mathcal{D}_\omega(\mathbb{R}^d)$ which we refer to as the *smear function* and which satisfies the following properties:

- $\sum_{k \in \mathcal{R}} \psi(\cdot - k) = 1$,
- $\psi = 1$ on $\text{supp } \varphi_j$ for $j < j_\mathcal{G}$,
- $\text{supp } \psi \subseteq B(0, R)$ with $R > 0$ small enough such that $B(0, R) \cap \mathcal{R} = \{0\}$.

The rescaled $\psi^\varepsilon := \psi(\varepsilon \cdot)$ satisfies the same properties on \mathcal{G}^ε . This allows us to define an extension operator \mathcal{E}^ε in the spirit of Lemma 2.6 as

$$\mathcal{E}^\varepsilon f := \mathcal{F}_{\mathbb{R}^d}^{-1}(\psi^\varepsilon \mathcal{F}f), \quad f \in \mathcal{S}'_\omega(\mathcal{G}^\varepsilon),$$

and as in Lemma 2.6 we can show that $\mathcal{E}^\varepsilon f \in C^\infty_\omega(\mathbb{R}^d) \cap \mathcal{S}'_\omega(\mathbb{R}^d)$ and $\mathcal{E}^\varepsilon f|_{\mathcal{G}^\varepsilon} = f$. Moreover, by the choice of ψ we have $\mathcal{E}^\varepsilon \Delta_j^{\mathcal{G}^\varepsilon} f = \Delta_j \mathcal{E}^\varepsilon f$ as long as $j < j_{\mathcal{G}^\varepsilon}$, and \mathcal{E}^ε is well behaved on the Besov spaces defined in Subsection 2.4.

Lemma 2.19. *For any $\alpha \in \mathbb{R}$, $p, q \in [1, \infty]$ and $\rho \in \boldsymbol{\rho}(\omega)$ for ω as in (14) or (15) the family of operators*

$$\mathcal{E}^\varepsilon: \mathcal{B}_{p,q}^\alpha(\mathcal{G}^\varepsilon, \rho) \longrightarrow \mathcal{B}_{p,q}^\alpha(\mathbb{R}^d, \rho)$$

is uniformly bounded in ε .

Proof. We have to estimate $\Delta_j \mathcal{E}^\varepsilon f$ for $j \lesssim j_{\mathcal{G}^\varepsilon}$. For $j < j_{\mathcal{G}^\varepsilon}$ we use $\Delta_j \mathcal{E}^\varepsilon f = \mathcal{E}^\varepsilon \Delta_j^{\mathcal{G}^\varepsilon} f$ and Lemma 2.16 to bound $\|\Delta_j \mathcal{E}^\varepsilon f\|_{L^p(\mathbb{R}^d, \rho)} = \|\varepsilon^{-d} \mathcal{F}_{\mathbb{R}^d} \psi(\varepsilon \cdot) * \Delta_j^{\mathcal{G}^\varepsilon} f\|_{L^p(\mathbb{R}^d, \rho)} \lesssim \|\Delta_j^{\mathcal{G}^\varepsilon} f\|_{L^p(\mathcal{G}^\varepsilon, \rho)}$ (where the convolution should be read as on page 14). For $j \sim j_{\mathcal{G}^\varepsilon}$ we have $\Delta_j \mathcal{E}^\varepsilon f = \Delta_j(\mathcal{E}^\varepsilon \sum_{i \sim j_{\mathcal{G}^\varepsilon}} \Delta_i^{\mathcal{G}^\varepsilon} f)$ which gives $\|\Delta_j \mathcal{E}^\varepsilon f\|_{L^p(\mathbb{R}^d, \rho)} \lesssim \sum_{i \sim j} \|\Delta_i^{\mathcal{G}^\varepsilon} f\|_{L^p(\mathcal{G}^\varepsilon, \rho)} \lesssim 2^{-j_{\mathcal{G}^\varepsilon} \alpha} \|f\|_{\mathcal{B}_{p,q}^\alpha(\mathcal{G}^\varepsilon, \rho)}$ with the same arguments as before. \square

In Section 4 we will often be given some functional $F(f_1, \dots, f_n)$ on discrete Besov functions that takes values in a discrete Besov space X (or some space constructed from it) and that satisfies a bound

$$\|F(f_1, \dots, f_n)\|_X \leq c(f_1, \dots, f_n). \quad (23)$$

We then say that the estimate (23) has the property (\mathcal{E}) (on X) if there is a “continuous version” \overline{F} of F and a continuous version \overline{X} of X and a sequence of constants $o_\varepsilon \rightarrow 0$ such that

$$\|\mathcal{E}^\varepsilon F(f_1, \dots, f_n) - \overline{F}(\mathcal{E}^\varepsilon f_1, \dots, \mathcal{E}^\varepsilon f_n)\|_{\overline{X}} \leq o_\varepsilon \cdot c(f_1, \dots, f_n) \quad (\mathcal{E})$$

In other words we can pull the operator \mathcal{E}^ε inside F without paying anything in the limit.

3 Discrete diffusion operators

3.1 Definitions

To construct a symmetric random walk on the lattice \mathcal{G}^ε that can reach every point (compare e.g. [LL10]) we choose a subset of “jump directions” $\{g_1, \dots, g_l\} \subseteq \mathcal{G} \setminus \{0\}$ such that $\mathbb{Z}g_1 + \dots + \mathbb{Z}g_l = \mathcal{G}$ and a map $\kappa: \{g_1, \dots, g_l\} \rightarrow (0, \infty)$. We then take as a rate for the jump by $\pm \varepsilon g_i$ the value $\kappa(g_i)/2\varepsilon^2$. In other words the generator of the random walk is

$$L^\varepsilon u(y) = \varepsilon^{-2} \sum_{e \in \{\pm g_i\}} \frac{\kappa(e)}{2} (u(y + \varepsilon e) - u(y)), \quad (24)$$

which converges (for u nice enough) to $Lu = \frac{1}{2} \sum_{i=1}^l \kappa(g_i) \partial_{g_i}^2 u$ as ε tends to 0, where ∂_{g_i} denotes the directional derivative. In the case $\mathcal{G} = \mathbb{Z}^d$ and $\kappa(e_i) = 1/d$ we obtain the simple random walk with limiting generator $L = \frac{1}{2d} \Delta$. We can reformulate (24) by introducing a signed measure

$$\mu = \kappa(g_1) \left(\frac{1}{2} \delta_{g_1} + \frac{1}{2} \delta_{-g_1} \right) + \dots + \kappa(g_l) \left(\frac{1}{2} \delta_{g_l} + \frac{1}{2} \delta_{-g_l} \right) - \sum_{i=1}^l \kappa(g_i) \delta_0,$$

which allows us to write $L^\varepsilon u = \varepsilon^{-2} \int_{\mathbb{R}^d} u(x + \varepsilon y) d\mu(y)$ and $Lu = \frac{1}{2} \int_{\mathbb{R}^d} \partial_y^2 u d\mu(y)$. In fact we will also allow the random walk to have infinite range:

Definition 3.1. *In the following μ is a finite, signed measure on a Bravais lattice \mathcal{G} such that*

- $\langle \text{supp } \mu \rangle = \mathcal{G}$,
- $\mu|_{\{0\}^c} \geq 0$,
- for any $\lambda > 0$ we have $\int_{\mathbb{R}^d} e^{\lambda \omega(x)} d|\mu|(x) < \infty$, where $|\mu|$ is the total variation of μ ,
- $\mu(A) = \mu(-A)$ for $A \subseteq \mathbb{R}^d$ and $\mu(\mathbb{R}^d) = 0$,

where $\langle \cdot \rangle$ denotes the subgroup generated by \cdot in $(\mathbb{R}^d, +)$ and where ω is of the form (15). We associate a norm on \mathbb{R}^d to μ which is given by

$$\|x\|_\mu^2 = \frac{1}{2} \int |x \cdot y|^2 d\mu(y).$$

Lemma 3.2. *The function $\|\cdot\|_\mu$ of Definition 3.1 is indeed a norm.*

Proof. The homogeneity is obvious and the triangle inequality follows from Minkowski's inequality. If $\|x\|_\mu = 0$ we have $x \cdot g = 0$ for all $g \in \text{supp } \mu$. Since $\langle \text{supp } \mu \rangle = \mathcal{G}$ we also have $x \cdot a_i = 0$ for the linearly independent vectors a_1, \dots, a_d from (2), which implies $x = 0$. \square

Definition 3.3. *For μ as above and \mathcal{G}^ε as in Definition 2.2 we set*

$$L^\varepsilon u(x) = \varepsilon^{-2} \int_{\mathbb{R}^d} u(x + \varepsilon y) d\mu(y)$$

for $u \in \mathcal{S}'_\omega(\mathcal{G}^\varepsilon)$ or $u \in \mathcal{S}'_\omega(\mathbb{R}^d)$, and

$$(Lu)(\varphi) := \frac{1}{2} \int_{\mathbb{R}^d} \partial_y^2 u d\mu(y)(\varphi) := \frac{1}{2} \int_{\mathbb{R}^d} (\partial_y^2 u)(\varphi) d\mu(y)$$

for $u \in \mathcal{S}'_\omega(\mathbb{R}^d)$ and $\varphi \in \mathcal{S}_\omega(\mathbb{R}^d)$. We write further $\mathcal{L}^\varepsilon, \mathcal{L}$ for the parabolic operators $\mathcal{L}^\varepsilon = \partial_t - L^\varepsilon$ and $\mathcal{L} = \partial_t - L$.

L^ε is nothing but the infinitesimal generator of a random walk with sub-exponential moments (Lemma A.7). By direct computation it can be checked that for $\mathcal{G} = \mathbb{Z}^d$ and with the extra condition $\int y_i y_j d\mu(y) = 2 \delta_{ij}$ we have the identities $\|\cdot\|_\mu = |\cdot|$ and $L = \Delta_{\mathbb{R}^d}$. In general L is an elliptic operator with constant coefficients,

$$Lu = \frac{1}{2} \int_{\mathbb{R}^d} \partial_y^2 u d\mu(y) = \frac{1}{2} \sum_{i,j} \int_{\mathbb{R}^d} y_i y_j d\mu(y) \cdot \partial^{ij} u =: \frac{1}{2} \sum_{i,j} a_{ij}^\mu \cdot \partial^{ij} \varphi,$$

where (a_{ij}^μ) is a symmetric matrix. The ellipticity condition follows from the relation $x \cdot (a_{ij}^\mu) x = \|x\|_\mu^2$ and the equivalence of norms on \mathbb{R}^d . In terms of regularity we expect therefore that L^ε behaves like the Laplacian when we work on discrete spaces.

Lemma 3.4. *We have for $\alpha \in \mathbb{R}$, $p \in [1, \infty]$ and $\rho \in \boldsymbol{\rho}(\omega)$ with ω as in (14) or (15)*

$$\|L^\varepsilon u\|_{C_p^{\alpha-2}(\mathcal{G}^\varepsilon, \rho)} \lesssim \|u\|_{C_p^\alpha(\mathcal{G}^\varepsilon, \rho)},$$

and for $\delta \in [0, 1]$

$$\|(L^\varepsilon - L)u\|_{C_p^{\alpha-2-\delta}(\mathbb{R}^d, \rho)} \lesssim \varepsilon^\delta \|u\|_{C_p^\alpha(\mathbb{R}^d, \rho)}.$$

Proof. We start with the first inequality. With $\overline{K}_j := \sum_{-1 \leq i \leq j_{\mathcal{G}^\varepsilon}: |i-j| \leq 1} K_i \in \mathcal{S}_\omega(\mathcal{G}^\varepsilon)$ we have $\Delta_j^{\mathcal{G}^\varepsilon} u = \overline{K}_j * \Delta_j^{\mathcal{G}^\varepsilon} u$. As on page 14 (and in Lemma A.2) we can write $\overline{K}_j = 2^{jd} \overline{\mathcal{K}}(2^j \cdot)$ with a smooth $\overline{\mathcal{K}} \in \mathcal{S}_\omega(\mathbb{R}^d)$ depending on the cases $j = -1$, $j \in \{0, \dots, j_{\mathcal{G}^\varepsilon} - 1\}$, or $j = j_{\mathcal{G}^\varepsilon}$. By putting derivatives on $2^j \overline{\mathcal{K}}(2^{jd} \cdot)$ in $\Delta_j^{\mathcal{G}^\varepsilon} u(x) = \sum_{k \in \mathcal{G}} 2^{jd} \overline{\mathcal{K}}(2^j(x-k)) \Delta_j^{\mathcal{G}^\varepsilon} u(k)$ we can differentiate $\Delta_j^{\mathcal{G}^\varepsilon} u$ and evaluate it on \mathbb{R}^d in the following.

Since μ integrates affine functions to zero we then have

$$\begin{aligned} \Delta_j^{\mathcal{G}^\varepsilon} L^\varepsilon u(x) &= \varepsilon^{-2} \int_{\mathbb{R}^d} d\mu(y) [\Delta_j^{\mathcal{G}^\varepsilon} u(x + \varepsilon y) - \Delta_j^{\mathcal{G}^\varepsilon} u(x) - \nabla(\Delta_j^{\mathcal{G}^\varepsilon} u)(x) \cdot \varepsilon y] \\ &= \int_{\mathbb{R}^d} d\mu(y) \int_0^1 d\zeta_1 \int_0^1 d\zeta_2 \ y \cdot \nabla^2(\Delta_j^{\mathcal{G}^\varepsilon} u)(x + \varepsilon \zeta_1 \zeta_2 y). \end{aligned}$$

Using $\int_0^1 d\zeta_1 \int_0^1 d\zeta_2 \int_{\mathbb{R}^d} d\mu(y) e^{\omega(\varepsilon \zeta_1 \zeta_2 y)} < \infty$ and (20) we see that we only have to show for $|\beta| = 2$ and $y \in \mathcal{G}$

$$\left\| \sum_{z \in \mathcal{G}^\varepsilon} |\mathcal{G}^\varepsilon| \partial^\beta \overline{K}_j(\cdot + \varepsilon \zeta_1 \zeta_2 y - z) \Delta_j^{\mathcal{G}^\varepsilon} u(z) \right\|_{L^p(\mathcal{G}^\varepsilon, \rho(\cdot + \varepsilon \zeta_1 \zeta_2 y))} \lesssim 2^{-2j} \|\Delta_j^{\mathcal{G}^\varepsilon} u\|_{L^p(\mathcal{G}, \rho)},$$

which follows from Lemma 2.16.

To show the second inequality we can similarly find a $\overline{K}_j = 2^{jd} \mathcal{K}(2^j \cdot) \in \mathcal{S}_\omega(\mathbb{R}^d)$ such that $\overline{K}_j *_{\mathbb{R}^d} \Delta_j u = \Delta_j u$ and proceeding as above we obtain

$$\Delta_j(L^\varepsilon - L)u = \int_0^1 d\zeta_1 \int_0^1 d\zeta_2 \int_{\mathbb{R}^d} d\mu(y) \int_{\mathbb{R}^d} dz \ y \cdot (\nabla^2 \overline{K}_j(\cdot + \varepsilon \zeta_1 \zeta_2 y - z) - \nabla^2 \overline{K}_j(\cdot - z)) y \Delta_j u(z),$$

which can be bounded by $2^{-j(\alpha-2)} \|u\|_{C_p^\alpha(\mathcal{G}^\varepsilon, \rho)}$ and $2^{-j(\alpha-3)} \varepsilon \|u\|_{C_p^\alpha(\mathcal{G}^\varepsilon, \rho)}$ and we obtain the second estimate by interpolation. \square

3.2 Semigroup estimates

In Fourier space L^ε can be represented by a Fourier multiplier

$$\widehat{L^\varepsilon u} = -l^\varepsilon \widehat{u},$$

where l^ε is given by

$$l^\varepsilon(x) = - \int_{\mathbb{R}^d} \frac{e^{i\varepsilon 2\pi xy}}{\varepsilon^2} d\mu(y) = \int_{\mathbb{R}^d} \frac{1 - \cos(\varepsilon 2\pi xy)}{\varepsilon^2} d\mu(y) = 2 \int_{\mathbb{R}^d} \frac{\sin^2(\varepsilon \pi xy)}{\varepsilon^2} d\mu(y), \quad (25)$$

where we used that μ is symmetric with $\mu(\mathbb{R}^d) = 0$. The following lemma shows that l^ε is well defined as a multiplier (i.e. $l^\varepsilon \in C_\omega^\infty(\widehat{\mathcal{G}^\varepsilon})$) and it is the backbone of the semigroup estimates shown below.

Lemma 3.5. *The function l^ε in (25) is an element of $\mathcal{S}_\omega(\widehat{\mathcal{G}}^\varepsilon) = C_\omega^\infty(\widehat{\mathcal{G}}^\varepsilon)$ and satisfies*

- $|\partial^\alpha l^\varepsilon(x)| \lesssim_\delta \varepsilon^{(|\alpha|-2) \vee 0} (1 + |x|^2) \delta^{|\alpha|} (\alpha!)^{1/\sigma}$ for any $\delta > 0$ and $\alpha \in \mathbb{N}^d$,
- $l^\varepsilon \gtrsim_K |\cdot|^2$ on every compact set $\varepsilon^{-1}K \subseteq \mathbb{R}^d$ with $K \cap \mathcal{R} = \{0\}$,

where $\sigma \in (0, 1)$ is the exponent of $\omega = |\cdot|^\sigma$.

Proof. We start by showing $|\partial^\alpha l^\varepsilon(x)| \lesssim_\delta \varepsilon^{(|\alpha|-2) \wedge 0} (1 + |x|^2) \delta^{|\alpha|} \alpha!$ which implies in particular $l^\varepsilon \in \mathcal{S}_\omega(\widehat{\mathcal{G}}^\varepsilon)$. We study derivatives with $|\alpha| = 0, 1$ first. We have

$$\begin{aligned} |l^\varepsilon(x)| &= 2 \left| \int_{\mathbb{R}^d} \frac{\sin^2(\varepsilon \pi x \cdot y)}{\varepsilon^2} d\mu(y) \right| \lesssim 2 \left| \int_{\mathbb{R}^d} \frac{\sin^2(\varepsilon \pi x \cdot y)}{|\varepsilon \pi x \cdot y|^2} |x \cdot y|^2 d\mu(y) \right| \\ &\lesssim \int |y|^2 d\mu(y) \cdot |x|^2 \lesssim |x|^2, \end{aligned}$$

and

$$|\partial^i l^\varepsilon(x)| \lesssim \int_{\mathbb{R}^d} \frac{|\sin(\varepsilon \pi x \cdot y)|}{|\varepsilon \pi x \cdot y|} |x| |y|^2 d\mu(y) \lesssim |x|.$$

For the higher derivatives we use that $\partial_x^\alpha e^{i\pi \varepsilon x \cdot y} = (i\pi \varepsilon)^{|\alpha|} y^\alpha e^{i\pi \varepsilon x \cdot y}$ which gives (where $C > 0$ denotes as usual a changing constant)

$$|\partial^\alpha l^\varepsilon(x)| \leq \varepsilon^{|\alpha|-2} \frac{1}{\varepsilon^2} C^{|\alpha|} \int_{\mathbb{R}^d} |y|^{|\alpha|} d\mu(y) \leq \varepsilon^{2-|\alpha|} C^{|\alpha|} \max_{x \in \mathbb{R}^d} (|x|^{|\alpha|} e^{-\lambda|x|^\sigma}) \int_{\mathbb{R}^d} e^{\lambda|y|^\sigma} d\mu(y)$$

for any $\lambda > 0$. Using $\max_{t \geq 0} t^a e^{-t} = a^a e^{-a}$ we end up with

$$|\partial^\alpha l^\varepsilon(x)| \lesssim \varepsilon^{2-|\alpha|} \frac{1}{\lambda^{|\alpha|/\sigma}} C^{|\alpha|} |\alpha|^{|\alpha|/\sigma} \lesssim \varepsilon^{2-|\alpha|} \frac{1}{\lambda^{|\alpha|/\sigma}} C^{|\alpha|} (\alpha!)^{1/\sigma},$$

and our first claim follows by choosing $\lambda^{1/\sigma} = C/\delta$.

It remains to show that $l^\varepsilon/|\cdot|^2 \gtrsim 1$ on $\varepsilon^{-1}K$, which is equivalent to $l^1/|\cdot|^2 \gtrsim 1$ on K . We start by finding the zeros of l^1 which, by periodicity can be reduced to finding all $x \in \widehat{\mathcal{G}}$ with $l^1(x) = 0$. But if $l^1(x) = 0$, then $y \cdot x \in \mathbb{Z}$ for any $y \in \text{supp } \mu$, which gives with $\langle \text{supp } \mu \rangle = \mathcal{G}$ that we must have $a_i \cdot x \in \mathbb{Z}$ for a_i as in (2). But since $x \in \widehat{\mathcal{G}}$ we have $x = x_1 \hat{a}_1 + \dots + x_d \hat{a}_d$ with $x_i \in [-1/2, 1/2)$ and \hat{a}_i as in (3). Consequently

$$x_i = x \cdot a_i \in \mathbb{Z} \cap [-1/2, 1/2) = \{0\},$$

and therefore $x = 0$. The zero set of l^1 is thus precisely the reciprocal lattice \mathcal{R} . By assumption $K \cap \mathcal{R} = \{0\}$ and it remains therefore to verify $l^1(x) \gtrsim |x|^2$ in an environment of 0 to finish the proof. Note that there is in fact a finite subset $V \subseteq \text{supp } \mu$ such that $\langle V \rangle = \mathcal{G}$ since only finitely many $y \in \text{supp } \mu$ are needed to generate a_1, \dots, a_d . We restrict ourselves to V :

$$l^1(x) = 2 \int_{\mathbb{R}^d} \sin^2(\pi x \cdot y) d\mu(y) \geq 2 \int_V \sin^2(\pi x \cdot y) d\mu(y)$$

For $x \in \widehat{\mathcal{G}} \setminus \{0\}$ small enough we can now bound $\int_V \sin^2(\pi x \cdot y) d\mu(y) \gtrsim \int_V |x \cdot y|^2 d\mu(y)$. The term on the right hand side defines a norm by the same arguments as in Lemma 3.2, and since it must be equivalent to $|\cdot|^2$ the proof is complete. \square

Using that $\mathcal{S}_\omega(\widehat{\mathcal{G}^\varepsilon}) = C_\omega^\infty(\widehat{\mathcal{G}^\varepsilon})$ is stable under composition we can now define the Fourier multiplier

$$e^{tL^\varepsilon} f := \mathcal{F}^{-1}(e^{-tl^\varepsilon} \mathcal{F} f)$$

for $f \in \mathcal{S}'_\omega(\mathcal{G}^\varepsilon)$ which gives the (weak) solution to the problem $\mathcal{L}^\varepsilon g = 0$, $g(0) = f$. The regularizing effect of the semigroup is estimated in the following proposition.

Proposition 3.6. *We have for $\alpha \in \mathbb{R}$, $\beta \geq 0$, $p \in [1, \infty]$, and $\rho \in \boldsymbol{\rho}(\omega)$ with ω as in (14), (15)*

$$\|e^{tL^\varepsilon} f\|_{\mathcal{C}_p^{\alpha+\beta}(\mathcal{G}^\varepsilon, \rho)} \lesssim t^{-\beta/2} \|f\|_{\mathcal{C}_p^\alpha(\mathcal{G}^\varepsilon, \rho)}, \quad (26)$$

$$\|e^{tL^\varepsilon} f\|_{\mathcal{C}_p^\beta(\mathcal{G}^\varepsilon, \rho)} \lesssim t^{-\beta/2} \|f\|_{L^p(\mathcal{G}^\varepsilon, \rho)}, \quad (27)$$

and for $\alpha \in (0, 2)$

$$\|(e^{tL^\varepsilon} - \text{Id})f\|_{L^p(\mathcal{G}^\varepsilon, \rho)} \lesssim t^{\alpha/2} \|f\|_{\mathcal{C}_p^\alpha(\mathcal{G}^\varepsilon, \rho)}, \quad (28)$$

uniformly on compact intervals $t \in [0, T]$.

Proof. We show the claim for ω as in (15), the arguments for ω as in (14) are similar but easier. As in the proof of Lemma 3.4 we have $\Delta_j^{\mathcal{G}^\varepsilon} e^{tL^\varepsilon} f = \overline{K}_j * \Delta_j^{\mathcal{G}^\varepsilon} e^{tL^\varepsilon} f$ with $\overline{K}_j = 2^{jd} \overline{\mathcal{K}}(2^j \cdot)$ for $\overline{\mathcal{K}} \in \mathcal{S}_\omega(\mathbb{R}^d)$, and we set $\varphi = \mathcal{F}_{\mathbb{R}^d} \overline{\mathcal{K}} \in \mathcal{D}_\omega(\mathbb{R}^d)$. Then we can rewrite for $x \in \mathcal{G}^\varepsilon$

$$\Delta_j e^{tL^\varepsilon} f(x) = \mathcal{F}_{\mathbb{R}^d}^{-1} \left(\varphi(2^{-j} \cdot) e^{-tl^\varepsilon} \mathcal{F} \Delta_j^{\mathcal{G}^\varepsilon} f \right) (x) = \sum_{k \in \mathcal{G}^\varepsilon} |\mathcal{G}^\varepsilon| 2^{jd} \mathcal{K}_j(t, 2^j(x-k)) \Delta_j f(k)$$

with $\mathcal{K}_j(t, x) = \int_{\mathbb{R}^d} dy e^{2\pi i xy} \varphi(y) e^{-tl^\varepsilon(2^j y)}$ and $\mathcal{F} \Delta_j^{\mathcal{G}^\varepsilon} f$ to be read as an \mathcal{R} -periodic distribution on \mathbb{R}^d (compare page 8).

Suppose we already know that for any $\lambda > 0$ and $x \in \mathcal{G}^\varepsilon$ the estimate

$$t^{\beta/2} |\mathcal{K}_j(t, x)| \lesssim e^{-\lambda|x|^\sigma} 2^{-j\beta} \quad (29)$$

holds. Then Young's inequality on \mathcal{G}^ε shows (26) and (27). Using Lemma 3.7 below we can reduce the task of proving (31) to the simpler problem of proving the polynomial bound

$$t^{\beta/2} |x_i|^n |\mathcal{K}_j(t, x)| \lesssim_\delta \delta^n C^n (n!)^{1/\sigma} 2^{-j\beta}, \quad (30)$$

with a constant $C > 0$ and an arbitrarily small $\delta > 0$.

To show (32) we assume that $2^j \varepsilon \leq 1$, otherwise we are dealing with the scale $2^j \sim \varepsilon^{-1}$ and the arguments below can be easily modified. Integration by parts gives us

$$|x_i|^n |\mathcal{K}_j(t, x)| = C^n \left| \int_{\mathbb{R}^d} dy e^{2\pi i xy} \partial_{y_i}^n \left(\varphi(y) e^{-tl^\varepsilon(2^j y)} \right) \right| \leq C^n \int_{\mathbb{R}^d} dy \left| \partial_{y_i}^n \left(\varphi(y) e^{-t2^{2j} l^{2\varepsilon}(y)} \right) \right|.$$

Now we have the estimates

$$\begin{aligned} |\partial_i^n \varphi(y)| &\lesssim \delta^n (n!)^{1/\sigma}, & |\partial^\alpha l^{\varepsilon 2^j}(y)| &\lesssim \delta^{|\alpha|} (\alpha!)^{1/\sigma} \\ \left| (2^j t)^{\beta/2} \left(e^{t2^{2j} \cdot} \right)^{(n)} (l^{2\varepsilon})(y) \right| &\lesssim n^{n/\sigma} \delta^n, \end{aligned}$$

where we used that $\varphi \in \mathcal{D}_\omega(\mathbb{R}^d)$ and Lemma 3.5 with the assumption $2^j \varepsilon \leq 1$. Together with Leibniz's and Faà-di Bruno's formulas and a lengthy but elementary calculation (32) follows and therefore also (31).

The last estimate (28) can be obtained as in the proof of Lemma [GP15b, Lemma 6.6] by using Lemma A.6. \square

Proof. We show the claim for ω as in (15), the arguments for (14) are similar but easier. As in the proof of Lemma 3.4 we have $\Delta_j^{\mathcal{G}^\varepsilon} e^{tL^\varepsilon} f = \bar{K}_j * \Delta_j^{\mathcal{G}^\varepsilon} e^{tL^\varepsilon} f$ with $\bar{K}_j = 2^{jd} \bar{K}(2^j \cdot)$ for $\bar{K} \in \mathcal{S}_\omega(\mathbb{R}^d)$ and could therefore in principal once more extend it to all of \mathbb{R}^d although we don't need to in this proof. We will write $\varphi = \mathcal{F}_{\mathbb{R}^d} \bar{K} \in \mathcal{D}_\omega(\mathbb{R}^d)$.

We can rewrite for $x \in \mathcal{G}^\varepsilon$ (or $x \in \mathbb{R}^d$)

$$\Delta_j e^{tL^\varepsilon} f(x) = \mathcal{F}_{\mathbb{R}^d}^{-1} \left(\varphi(2^{-j} \cdot) e^{-tl^\varepsilon} \cdot \mathcal{F} \Delta_j^{\mathcal{G}^\varepsilon} f \right) (x) = \sum_{k \in \mathcal{G}^\varepsilon} |\mathcal{G}^\varepsilon| 2^{jd} \mathcal{K}_j(t, 2^j(x-k)) \Delta_j f(x')$$

with $\mathcal{K}_j(t, x) = \int_{\mathbb{R}^d} dy e^{2\pi i xy} \varphi(y) e^{-tl^\varepsilon(2^j y)}$ and $\mathcal{F} \Delta_j^{\mathcal{G}^\varepsilon} f$ to be read as an \mathcal{R} -periodic distribution on \mathbb{R}^d (compare page 8).

Suppose we already know that for any $\lambda > 0$ and $x \in \mathcal{G}^\varepsilon$ (or $x \in \mathbb{R}$) the estimate

$$t^{\beta/2} |\mathcal{K}_j(t, x)| \lesssim e^{-\lambda|x|^\sigma} 2^{-j\beta}. \quad (31)$$

holds. An application of Young's inequality on \mathcal{G}^ε then shows (26) and (27). Using Lemma 3.7 below we can reduce the task to prove (31) to the simpler problem of proving a polynomial bound:

$$t^{\beta/2} |x_i|^n |\mathcal{K}_j(t, x)| \lesssim_\delta \delta^n C^n (n!)^{1/\sigma} 2^{-j\beta}, \quad (32)$$

with a constant $C > 0$ and an arbitrarily small $\delta > 0$, because a Taylor expansion of $e^{\lambda|x|^\sigma}$ then gives the sub-exponential bound (31) (compare the proof of Lemma A.1).

We assume that $2^j \varepsilon \leq 1$, if this is not true we are dealing with the scale $2^j \sim \varepsilon^{-1}$ and the arguments below can be easily modified.

Integration by parts gives us

$$|x_i|^n |\mathcal{K}_j(t, x)| = C^n \left| \int_{\mathbb{R}^d} dy e^{2\pi i xy} \partial_{y_i}^n \left(\varphi(y) e^{-tl^\varepsilon(2^j y)} \right) \right| \leq C^n \int_{\mathbb{R}^d} dy \left| \partial_{y_i}^n \left(\varphi(y) e^{-t2^{2j} l^{2j\varepsilon}(y)} \right) \right|.$$

We have the estimates

$$\begin{aligned} |\partial_i^n \varphi(y)| &\lesssim \delta^n (n!)^{1/\sigma}, \quad |\partial^\alpha l^{\varepsilon 2^j}(y)| \lesssim \delta^{|\alpha|} (\alpha!)^{1/\sigma} \\ \left| (2^j t)^{\beta/2} \left(e^{t2^{2j} \cdot} \right)^{(n)} (l^{2j\varepsilon})(y) \right| &\lesssim n^{n/\sigma} \delta^n \end{aligned}$$

where we used $\varphi \in \mathcal{D}_\omega(\mathbb{R}^d)$ and Lemma 3.5 with the assumption $2^j \varepsilon \leq 1$. An application of Leibniz's and Faà-di Bruno's formula then shows after a bit lengthy calculation (32) and therefore (31).

The last estimate can be obtained as in the proof of Lemma [GP15b, Lemma 6.6] by using Lemma A.6. \square

Lemma 3.7. *Let $g: \mathbb{R}^d \rightarrow \mathbb{R}$, $\sigma > 0$ and $K > 0$. Suppose for any $\delta > 0$ there is a $C_\delta > 0$ such that for all $z \in \mathbb{R}^d$, $l \geq 0$ and $i = 1, \dots, d$*

$$|z_i^l g(z)| \lesssim_\delta \delta^l C_\delta^l (l!)^{1/\sigma} K.$$

It then holds for any $\lambda > 0$ and $z \in \mathbb{R}^d$

$$|g(z)| \lesssim_\lambda K e^{-\lambda|z|^\sigma}.$$

Proof. This follows ideas from [MW15, Proposition A.2]. Without loss of generality we can assume $|z| > 1$ (otherwise we get the required estimate by taking $l = 0$). Note that we have $|z|^l \leq C^l \sum_{i=1}^d |z_i|^l$ where $C > 0$ denotes a constant that changes from line to line. Consequently, Stirling's formula gives

$$\begin{aligned} |e^{\lambda|z|^\sigma} g(z)| &= \left| \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} |z|^{\sigma k} g(z) \right| \lesssim \sum_{k=0}^{\infty} \frac{\lambda^k C^k}{k^k} |z|^{[k\sigma]} |g(z)| \lesssim \sum_{k=0}^{\infty} \frac{\lambda^k C^k}{k^k} \sum_{i=1}^d |z_i|^{[k\sigma]} |g(z)| \\ &\lesssim K \sum_{k=0}^{\infty} \frac{\lambda^k C^k \delta^{k\sigma}}{k^k} [k\sigma]^{[k\sigma]/\sigma} \lesssim K \sum_{k=0}^{\infty} \frac{\lambda^k C^k \delta^{k\sigma}}{k^k} k^{k+1/\sigma} \lesssim K \sum_{k=0}^{\infty} \lambda^k C^k \delta^{k\sigma} \lesssim_\lambda K, \end{aligned}$$

where we chose $\delta > 0$ small enough in the last step. \square

3.3 Schauder estimates

We will follow here closely [GP15b] and introduce time-weighted parabolic spaces $\mathcal{L}_{p,T}^{\gamma,\alpha}$ that interplay nicely with the semigroup e^{tL^ε} .

Definition 3.8. Given $\gamma \geq 0$, $T > 0$ and a family of increasing normed spaces $X = (X(s))_{s \in [0,T]}$ we define the space

$$\mathcal{M}_T^\gamma X := \left\{ f: [0, T] \rightarrow X(T) \mid \|f\|_{\mathcal{M}_T^\gamma X} = \sup_{t \in [0, T]} \|t^\gamma f(t)\|_{X(t)} < \infty \right\},$$

and for $\alpha > 0$

$$C_T^\alpha X = \{f \in C([0, T], X(T)) \mid \|f\|_{C_T^\alpha X} < \infty\},$$

where

$$\|f\|_{C_T^\alpha X} = \sup_{t \in [0, T]} \|f(t)\|_{X(t)} + \sup_{0 \leq s \leq t \leq T} \frac{\|f(s) - f(t)\|_{X(t)}}{|s - t|^\alpha}.$$

For a lattice \mathcal{G} , $\gamma \geq 0$, $T > 0$, $\alpha \geq 0$ and a pointwise decreasing map $\rho: [0, T] \ni t \mapsto \rho(t) \in \boldsymbol{\rho}(\omega)$ we set

$$\mathcal{L}_{p,T}^{\gamma,\alpha}(\mathcal{G}, \rho) = \left\{ f: [0, T] \rightarrow \mathcal{S}'_\omega(\mathcal{G}) \mid \|f\|_{\mathcal{L}_{p,T}^{\gamma,\alpha}(\mathcal{G}, \rho)} < \infty \right\},$$

where

$$\|f\|_{\mathcal{L}_{p,T}^{\gamma,\alpha}(\mathcal{G}, \rho)} = \|t \mapsto t^\gamma f(t)\|_{C_T^{\alpha/2} L^p(\mathcal{G}, \rho)} + \|f\|_{\mathcal{M}_T^\gamma C_p^\alpha(\mathcal{G}, \rho)}.$$

Standard arguments show that if X is a sequence of increasing Banach spaces with decreasing norms, all the spaces in the previous definition are in fact complete in their (semi-)norms.

The Schauder estimates for the operator

$$I^\varepsilon f(t) = \int_0^t e^{(t-s)L^\varepsilon} f(s) \, ds \quad (33)$$

and the semigroup (e^{tL^ε}) in the time-weighted setup are summarized in the following lemma, for which we recall that $p^\kappa(x) = (1 + |x|)^{-\kappa}$ and $e_{l+t}^\sigma(x) = e^{-(l+t)(1+|x|)^\sigma}$. The notation $\mathcal{L}_{p,T}^{\gamma,\alpha}(\mathcal{G}, e_l^\sigma)$ means that we take the time-dependent weight $(e_{l+t}^\sigma)_{t \in [0, T]}$, while $e_l^\sigma p^\kappa$ stands for the time-dependent weight $(e_{l+t}^\sigma p^\kappa)_{t \in [0, T]}$.

Lemma 3.9. *Let $\alpha \in (0, 2), \gamma \in [0, 1], p \in [1, \infty], \sigma \in (0, 1)$ and $T > 0$. If $\beta \in \mathbb{R}$ is such that $(\alpha + \beta)/2 \in [0, 1]$, then we have uniformly in ε*

$$\|s \mapsto e^{sL^\varepsilon} f_0\|_{\mathcal{L}_{p,T}^{(\alpha+\beta)/2, \alpha}(\mathcal{G}^\varepsilon, e_l^\sigma)} \lesssim \|f_0\|_{C_p^{-\beta}(\mathcal{G}^\varepsilon, e_l^\sigma)}, \quad (34)$$

and if $\kappa \geq 0$ is such that $\gamma + \kappa/\sigma \in [0, 1]$, $\alpha + 2\kappa/\sigma \in (0, 2)$ also

$$\|I^\varepsilon f\|_{\mathcal{L}_{p,T}^{\gamma, \alpha}(\mathcal{G}^\varepsilon, e_l^\sigma)} \lesssim \|f\|_{\mathcal{M}_T^\gamma C_p^{\alpha+2\kappa/\sigma-2}(\mathcal{G}^\varepsilon, e_l^\sigma p^\kappa)}. \quad (35)$$

Proof. The proof is along the lines of Lemma 6.6 in [GP15b] with the use of the simple estimate

$$e_{l+t}^\sigma \lesssim \frac{1}{|t-s|^{\kappa/\sigma}} p^\kappa e_{l+s}^\sigma, \quad t \geq s,$$

which is similar to an inequality from the proof of Proposition 4.2 in [HL15] and the reason for the appearance of the term $2\kappa/\sigma$ in the lower estimate (the factor 2 comes from the parabolic scaling). We need $\gamma + \kappa/\sigma \in [0, 1]$ so that the singularity $|t-s|^{-\gamma-\kappa/\sigma}$ is integrable on $[0, t]$. \square

For the comparison of the parabolic spaces $\mathcal{L}_{p,T}^{\gamma, \alpha}$ the following lemma will be convenient.

Lemma 3.10. *For $\alpha \in (0, 2), \gamma \in (0, 1), \varepsilon \in [0, \alpha \wedge 2\gamma), p \in [1, \infty], T > 0$ and a pointwise decreasing $\mathbb{R}_+ \ni s \mapsto \rho(s) \in \boldsymbol{\rho}(\omega)$ we have*

$$\|f\|_{\mathcal{L}_{p,T}^{\gamma-\varepsilon/2, \alpha-\varepsilon}(\mathcal{G}^\varepsilon, \rho)} \lesssim \|f\|_{\mathcal{L}_{p,T}^{\gamma, \alpha}(\mathcal{G}^\varepsilon, \rho)},$$

and for $\gamma \in [0, 1]$ and $\varepsilon \in (0, \alpha)$

$$\|f\|_{\mathcal{L}_{p,T}^{\gamma, \alpha-\varepsilon}(\mathcal{G}^\varepsilon, \rho)} \lesssim \mathbf{1}_{\gamma=0} \|f(0)\|_{C_p^{\alpha-\varepsilon}(\mathcal{G}^\varepsilon, \rho)} + T^{\varepsilon/2} \|f\|_{\mathcal{L}_{p,T}^{\gamma, \alpha}(\mathcal{G}^\varepsilon, \rho)}.$$

Proof. The first estimate is proved as in [GP15b, Lemma 6.8]. For $\gamma = 0$ the proof of the second inequality works as in Lemma 2.11 of [GP15b]. The general case follows from the fact that $f \in \mathcal{L}_{p,T}^{\gamma, \alpha}$ if and only if $t \mapsto t^\gamma f \in \mathcal{L}_{p,T}^{0, \alpha}$. \square

4 Paracontrolled analysis on Bravais lattices

4.1 Discrete Paracontrolled Calculus

Given two distributions $f_1, f_2 \in \mathcal{S}'(\mathbb{R}^d)$ Bony [Bon81] defines their *paraproduct* by

$$f_1 < f_2 := \sum_{1 \leq j_2} \sum_{-1 \leq j_1 < j_2 - 1} \Delta_{j_1} f_1 \cdot \Delta_{j_2} f_2 = \sum_{1 \leq j_2} S_{j_2-1} f_1 \cdot \Delta_{j_2} f_2,$$

which turns out to always be a well-defined expression. However, to make sense of the product $f_1 f_2$ it is not sufficient to consider $f_1 < f_2$ and $f_1 > f_2 := f_2 < f_1$, we also have to take into account the *resonant term* [GIP15]

$$f_1 \circ f_2 := \sum_{-1 \leq j_1, j_2: |j_1 - j_2| \leq 1} \Delta_{j_1} f_1 \cdot \Delta_{j_2} f_2,$$

which can in general only be defined under compatible regularity conditions such as $f_1 \in \mathcal{C}_\infty^\alpha(\mathbb{R}^d)$, $f_2 \in \mathcal{C}_\infty^\beta(\mathbb{R}^d)$ with $\alpha + \beta > 0$ (see e.g. [BCD11] or [GIP15, Lemma 2.1]). If these conditions are satisfied we decompose $f_1 f_2 = f_1 < f_2 + f_1 > f_2 + f_1 \circ f_2$. Bony's construction can easily be adapted to a discrete and weighted setup, where of course we have no problem in making sense of pointwise products but we are interested in uniform estimates.

Definition 4.1. Given ω as in (14) or (15) and $f_1, f_2 \in \mathcal{S}'_\omega(\mathbb{R}^d)$ we define the discrete paraproduct

$$f_1 <^{\mathcal{G}} f_2 := \sum_{1 \leq j_2 \leq j_{\mathcal{G}}} \sum_{-1 \leq j_1 < j_2 - 1} \Delta_{j_1}^{\mathcal{G}} f_1 \cdot \Delta_{j_2}^{\mathcal{G}} f_2, \quad (36)$$

and we also write $f_1 >^{\mathcal{G}} f_2 := f_2 <^{\mathcal{G}} f_1$. The discrete resonant product is

$$f_1 \circ^{\mathcal{G}} f_2 := \sum_{1 \leq j_1, j_2 \leq j_{\mathcal{G}}, |j_1 - j_2| \leq 1} \Delta_{j_1}^{\mathcal{G}} f_1 \cdot \Delta_{j_2}^{\mathcal{G}} f_2. \quad (37)$$

If there is no risk for confusion we may drop the index \mathcal{G} on $<, >$, and \circ .

In contrast to the continuous theory $\circ^{\mathcal{G}}$ is well defined without any further restrictions since it only involves a finite sum. All the estimates that are known from the continuous theory carry over.

Lemma 4.2. Given $\rho_1, \rho_2 \in \rho(\omega)$ and $p \in [1, \infty]$ we have the bounds:

- For any $\alpha_2 \in \mathbb{R}$

$$\|f_1 < f_2\|_{C_p^{\alpha_2}(\mathcal{G}^\varepsilon, \rho_1, \rho_2)} \lesssim \|f_1\|_{L^\infty(\mathcal{G}^\varepsilon, \rho_1)} \|f_2\|_{C_p^{\alpha_2}(\mathcal{G}^\varepsilon, \rho_2)} \wedge \|f_1\|_{L^p(\mathcal{G}^\varepsilon, \rho_1)} \|f_2\|_{C_\infty^{\alpha_2}(\mathcal{G}^\varepsilon, \rho_2)},$$

- for any $\alpha_1 < 0, \alpha_2 \in \mathbb{R}$

$$\|f_1 < f_2\|_{C_p^{\alpha_1 + \alpha_2}(\mathcal{G}^\varepsilon, \rho_1, \rho_2)} \lesssim \|f_1\|_{C_p^{\alpha_1}(\mathcal{G}^\varepsilon, \rho_1)} \|f_2\|_{C_\infty^{\alpha_2}(\mathcal{G}^\varepsilon, \rho_2)} \wedge \|f_1\|_{C_\infty^{\alpha_1}(\mathcal{G}^\varepsilon, \rho_1)} \|f_2\|_{C_p^{\alpha_2}(\mathcal{G}^\varepsilon, \rho_2)},$$

- for any $\alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha_1 + \alpha_2 > 0$

$$\|f_1 \circ f_2\|_{C_p^{\alpha_1 + \alpha_2}(\mathcal{G}^\varepsilon, \rho_1, \rho_2)} \lesssim \|f_1\|_{C_p^{\alpha_1}(\mathcal{G}^\varepsilon, \rho_1)} \|f_2\|_{C_\infty^{\alpha_2}(\mathcal{G}^\varepsilon, \rho_2)} \wedge \|f_1\|_{C_p^{\alpha_1}(\mathcal{G}^\varepsilon, \rho_1)} \|f_2\|_{C_\infty^{\alpha_2}(\mathcal{G}^\varepsilon, \rho_2)},$$

where all involved constants only depend on \mathcal{G} but not on ε . All estimates have the property (\mathcal{E}) if the regularity on the left hand side is lowered by an arbitrary $\kappa > 0$.

Proof. The proof of the estimates follows along the lines of [GIP15, Lemma 2.1]) which in turn is taken from [BCD11, Theorem 2.82, Theorem 2.85]. To check the (\mathcal{E}) -property we recall that $\mathcal{E}^\varepsilon = \psi(\varepsilon D)$ with $\psi(\varepsilon \cdot) = 1$ in some ball of order $\varepsilon^{-1} \approx 2^{-j_{\mathcal{G}^\varepsilon}}$ inside $\widehat{\mathcal{G}^\varepsilon}$. We thus have by the spectral support properties of the paraproduct

$$\Delta_i(\mathcal{E}^\varepsilon(f_1 <^{\mathcal{G}^\varepsilon} f_2) - \mathcal{E}^\varepsilon f_1 <^{\mathcal{G}^\varepsilon} f_2) = \mathbf{1}_{i \sim j_{\mathcal{G}^\varepsilon}} \left(\Delta_i \mathcal{E}^\varepsilon \left(\sum_{j \sim i} S_{j-1}^{\mathcal{G}^\varepsilon} f_1 \Delta_j^{\mathcal{G}^\varepsilon} f_2 \right) - \Delta_i \left(\sum_{j \sim i} S_{j-1} f_1 \Delta_j f_2 \right) \right).$$

Together with Lemma 2.16 this gives for the first two estimates the bounds $\mathbf{1}_{i \sim j_{\mathcal{G}^\varepsilon}} 2^{-i\alpha_2} \lesssim 2^{-i(\alpha_2 - \kappa)} \varepsilon^\kappa$ and $\mathbf{1}_{i \sim j_{\mathcal{G}^\varepsilon}} 2^{-i(\alpha_1 + \alpha_2)} \lesssim 2^{-i(\alpha_1 + \alpha_2 - \kappa)} \varepsilon^\kappa$. For the third case we obtain by similar arguments for $\Delta_i(\mathcal{E}^\varepsilon(f_1 \circ^{\mathcal{G}^\varepsilon} f_2) - \mathcal{E}^\varepsilon f_1 \circ^{\mathcal{G}^\varepsilon} f_2)$ the bound

$$\sum_{j: i \lesssim j \sim j_{\mathcal{G}^\varepsilon}} 2^{-j(\alpha_1 + \alpha_2)} \lesssim \mathbf{1}_{i \lesssim j_{\mathcal{G}^\varepsilon}} 2^{-j_{\mathcal{G}^\varepsilon}(\alpha_1 + \alpha_2)} \lesssim 2^{-i(\alpha_1 + \alpha_2 - \kappa)} \varepsilon^\kappa,$$

for $\kappa > 0$ small enough such that $\alpha_1 + \alpha_2 - \kappa > 0$. □

The main observation of [GIP15] is that if the regularity condition $\alpha_1 + \alpha_2 > 0$ is not satisfied, then it may still be possible to make sense of $f_1 \circ f_2$ as long as f_1 can be written as a paraproduct plus a smoother remainder. The main lemma which makes this possible is an estimate for a certain commutator. The discrete version of the commutator is defined as

$$C^{\mathcal{G}}(f_1, f_2, f_3) := (f_1 \triangleleft^{\mathcal{G}} f_2) \circ^{\mathcal{G}} f_3 - f_1(f_2 \circ^{\mathcal{G}} f_3).$$

If there is no risk for confusion we may drop the index \mathcal{G} on C .

Lemma 4.3. ([GP15c, Lemma 14]) *Given $\rho_1, \rho_2, \rho_3 \in \boldsymbol{\rho}(\omega)$, $p \in [1, \infty]$ and $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ with $\alpha_1 + \alpha_2 + \alpha_3 > 0$ and $\alpha_2 + \alpha_3 \neq 0$ we have*

$$\|C^{\mathcal{G}}(f_1, f_2, f_3)\|_{C_p^{\alpha_2+\alpha_3}(\mathcal{G}^\varepsilon, \rho_1 \rho_2 \rho_3)} \lesssim \|f_1\|_{C_p^{\alpha_1}(\mathcal{G}^\varepsilon, \rho_1)} \|f_2\|_{C_\infty^{\alpha_2}(\mathcal{G}^\varepsilon, \rho_2)} \|f_3\|_{C_\infty^{\alpha_3}(\mathcal{G}^\varepsilon, \rho_3)}.$$

Further, property (\mathcal{E}) holds for C if the regularity on the left hand side is reduced by an arbitrary $\kappa > 0$.

Proof. The proof of the estimates works line-by-line as in [GP15c, Lemma 14] and the (\mathcal{E}) -property follows as in Lemma 4.2 by exploiting that $\psi(\varepsilon^{-1} \cdot) = 1$ on a ball of order ε^{-1} . \square

4.2 The modified paraproduct

It will be useful to define a lattice version of the *modified paraproduct* \ll that was introduced in [GIP15] and also used in [GP15b, CGP17].

Definition 4.4. *Fix a function $\varphi \in C_c^\infty((0, \infty); \mathbb{R}_+)$ such that $\int_{\mathbb{R}} \varphi(s) ds = 1$ and define*

$$Q_i f(t) := \int_{-\infty}^t 2^{2id} \varphi(2^{2i}(t-s)) f(s \vee 0) ds, \quad i \geq -1.$$

We then set

$$f_1 \ll^{\mathcal{G}} f_2 := \sum_{-1 \leq j_1, j_2 \leq j_{\mathcal{G}}: j_1 < j_2 - 1} Q_{j_2} \Delta_{j_1}^{\mathcal{G}} f_1 \cdot \Delta_{j_2}^{\mathcal{G}} f_2$$

for $f_1, f_2: \mathbb{R}_+ \rightarrow \mathcal{S}'_{\omega}(\mathcal{G})$ where this is well defined. We may drop the index \mathcal{G} if there is no risk for confusion.

As in [GP15b] we silently identify f_1 in $f_1 \ll f_2$ with $t \mapsto f(t) \mathbf{1}_{t>0}$ if $f_1 \in \mathcal{M}_T^{\gamma} \mathcal{C}_p^{\alpha}$. Once more the generalization to the continuous case $f_1, f_2: \mathbb{R}_+ \rightarrow \mathcal{S}'_{\omega}(\mathbb{R}^d)$ is obvious. The modified paraproduct allows for similar estimates as in Lemma 4.2.

Lemma 4.5. *Let $\beta \in \mathbb{R}$, $p \in [1, \infty]$, $\gamma \in [0, 1)$, $t > 0$, $\alpha < 0$ and let $\rho_1, \rho_2: \mathbb{R}_+ \rightarrow \boldsymbol{\rho}(\omega)$ with ρ_1 pointwise decreasing. Then*

$$t^{\gamma} \|f \ll g(t)\|_{C_p^{\alpha+\beta}(\mathcal{G}^\varepsilon, \rho_1(t) \rho_2(t))} \lesssim \|f\|_{\mathcal{M}_t^{\gamma} \mathcal{C}_p^{\alpha}(\mathcal{G}^\varepsilon, \rho_1)} \|g(t)\|_{C_\infty^{\beta}(\mathcal{G}^\varepsilon, \rho_2(t))} \wedge \|f\|_{\mathcal{M}_t^{\gamma} \mathcal{C}_\infty^{\alpha}(\mathcal{G}^\varepsilon, \rho_1)} \|g(t)\|_{C_p^{\beta}(\mathcal{G}^\varepsilon, \rho_2(t))}$$

and

$$t^{\gamma} \|f \ll g(t)\|_{C_p^{\beta}(\mathcal{G}^\varepsilon, \rho_1(t) \rho_2(t))} \lesssim \min\{\|f\|_{\mathcal{M}_t^{\gamma} L^p(\mathcal{G}^\varepsilon, \rho_1)} \|g(t)\|_{C_\infty^{\beta}(\mathcal{G}^\varepsilon, \rho_2)} \wedge \|f\|_{\mathcal{M}_t^{\gamma} L^\infty(\mathcal{G}^\varepsilon, \rho_1)} \|g(t)\|_{C_p^{\beta}(\mathcal{G}^\varepsilon, \rho_2(t))}\}.$$

Both estimates have the property (\mathcal{E}) if the regularity on the left hand side is decreased by an arbitrary $\kappa > 0$.

Proof. The proof is the same as for [GP15b, Lemma 6.4]. Property (\mathcal{E}) is shown as in Lemma 4.2. \square

We further have an estimate in terms of the parabolic spaces $\mathcal{L}_{p,T}^{\gamma,\alpha}(\mathcal{G}, \rho)$ that were introduced in Definition 3.8.

Lemma 4.6. *We have for $\alpha \in (0, 2)$, $p \in [1, \infty]$, $\gamma \in [0, 1)$ and $\rho_1, \rho_2: \mathbb{R}_+ \rightarrow \rho(\omega)$, pointwise decreasing in s , the estimate*

$$\|f \ll g\|_{\mathcal{L}_{p,T}^{\gamma,\alpha}(\mathcal{G}^\varepsilon, \rho_1 \rho_2)} \lesssim \|f\|_{\mathcal{L}_{p,T}^{\gamma,\delta}(\mathcal{G}^\varepsilon, \rho_1)} (\|g\|_{C_T C_\infty^\alpha(\mathcal{G}^\varepsilon, \rho_2)} + \|\mathcal{L}^\varepsilon g\|_{C_T C_\infty^{\alpha-2}(\mathcal{G}^\varepsilon, \rho_2)})$$

for any $\delta > 0$ and any diffusion operator \mathcal{L}^ε as in Definition 3.3.

Proof. The proof is as in [GP15b, Lemma 6.7] and uses Lemma 4.7 below. \square

The main advantage of the modified paraproduct \ll on \mathbb{R}^d is its commutation property with the heat kernel $\partial_t - \Delta$ (or $\mathcal{L} = \partial_t - L$) which is essential for the Schauder estimates for paracontrolled distributions, compare also Subsection 5.2 below. In the following we state the corresponding results for Bravais lattices.

Lemma 4.7. *For $\alpha \in (0, 2)$, $\beta \in \mathbb{R}$, $p \in [1, \infty]$, $\gamma \in [0, 1)$ and $\rho_1, \rho_2: \mathbb{R}_+ \rightarrow \rho(\omega)$, with ρ_1 pointwise decreasing, we have for $t > 0$*

$$t^\gamma \|(f \ll g - f \langle g \rangle)(t)\|_{C_p^{\alpha+\beta}(\mathcal{G}^\varepsilon, \rho_1(t) \rho_2(t))} \lesssim \|f\|_{\mathcal{L}_{p,t}^{\gamma,\alpha}(\mathcal{G}^\varepsilon, \rho_1)} \|g(t)\|_{C_\infty^\beta(\mathcal{G}^\varepsilon, \rho_2(t))}$$

and

$$t^\gamma \|\mathcal{L}^\varepsilon(f \ll g) - f \ll \mathcal{L}^\varepsilon g(t)\|_{C_p^{\alpha+\beta-2}(\mathcal{G}^\varepsilon, \rho_1(t) \rho_2(t))} \lesssim \|f\|_{\mathcal{L}_{p,t}^{\gamma,\alpha}(\mathcal{G}^\varepsilon, \rho_1)} \|g(t)\|_{C_\infty^\beta(\mathcal{G}^\varepsilon, \rho_2(t))}.$$

where $\mathcal{L}^\varepsilon = \partial_t - L^\varepsilon$ is a discrete diffusion operator as in Definition 3.3

Proof. The proof is almost the same as in [GP15b, Lemma 6.5] with the only difference that the application of the “product rule” of \mathcal{L}^ε for the second bound does not yield a term $-2\nabla f \ll \nabla g$ but an object that is slightly more complicated and which we bound in Lemma A.8. \square

5 Weak universality of PAM on \mathbb{R}^2

With the structures and estimates from Sections 2-4 at hand we are now able to analyse stochastic models on unbounded lattices using paracontrolled techniques. As an example, we prove the weak universality result for the linear parabolic Anderson model that we discussed in the introduction. For $F \in C^2(\mathbb{R}; \mathbb{R})$ with $F(0) = 0$ and bounded second derivative we consider the equation

$$\mathcal{L}^1 v^\varepsilon = F(v^\varepsilon) \eta^\varepsilon, \quad v^\varepsilon(0) = |\mathcal{G}|^{-1} \mathbf{1}_{=0} \quad (38)$$

on $\mathbb{R}_+ \times \mathcal{G}$, where \mathcal{G} is a two-dimensional Bravais lattice, \mathcal{L}^1 is some discrete diffusion operator on \mathcal{G} as in Section 3, and $(\eta^\varepsilon(z))_{z \in \mathcal{G}} \in \mathcal{S}'_\omega(\mathcal{G})$ is a family of independent (not necessarily identically distributed) random variables with uniformly bounded moments of order $p_\xi > 14$ and such that

$$\mathbb{E}[\eta^\varepsilon] = -F'(0) c^\varepsilon \varepsilon^2, \quad \text{Var}(\eta^\varepsilon) = \frac{1}{|\mathcal{G}|} \varepsilon^2,$$

where $c^\varepsilon > 0$ is a constant of order $O(|\log \varepsilon|)$ which we will fix in Subsection 5.1 below. Note that η^ε is of order $O(\varepsilon)$ while its expectation is of order $O(\varepsilon^2 |\log \varepsilon|)$, so we are considering a small shift away from the “critical” expectation 0.

We are interested in the behaviour of (38) for large scales in time and space. Setting $u^\varepsilon(t, x) := \varepsilon^{-2} v(\varepsilon^{-2} t, \varepsilon^{-1} x)$ and $\xi^\varepsilon(x) := \varepsilon^{-2} (\eta^\varepsilon(\varepsilon^{-1} x) + F'(0) c^\varepsilon \varepsilon^2)$ modifies the problem to

$$\mathcal{L}^\varepsilon u^\varepsilon = F^\varepsilon(u^\varepsilon)(\xi^\varepsilon - F'(0) c^\varepsilon), \quad u^\varepsilon(0) = |\mathcal{G}^\varepsilon|^{-1} \mathbf{1}_{=0}, \quad (39)$$

where $u^\varepsilon: \mathbb{R}_+ \times \mathcal{G}^\varepsilon \rightarrow \mathbb{R}$ on refining lattices \mathcal{G}^ε in $d = 2$ and where $F^\varepsilon = \varepsilon^{-2} F(\varepsilon^2 \cdot)$. The potential $(\xi^\varepsilon(x))_{x \in \mathcal{G}^\varepsilon}$ is scaled such that it satisfies for $x \in \mathcal{G}^\varepsilon$

- $\mathbb{E}[\xi^\varepsilon(x)] = 0$,
- $\mathbb{E}[|\xi^\varepsilon(x)|^2] = |\mathcal{G}^\varepsilon|^{-1} = |\mathcal{G}|^{-1} \varepsilon^{-2}$,
- $\sup_{z \in \mathcal{G}^\varepsilon} \mathbb{E}[|\xi^\varepsilon(z)|^{p_\xi}] \lesssim \varepsilon^{-p_\xi}$ for some $p_\xi > 14$.

Consequently, $\mathcal{E}^\varepsilon \xi^\varepsilon$ converges in distribution to the two-dimensional space white noise. In Theorem 5.10 we show that $\mathcal{E}^\varepsilon u^\varepsilon$ converges in distribution to the solution u of the linear parabolic Anderson model on \mathbb{R}^2 ,

$$\mathcal{L}u = F'(0)u(\xi - F'(0)\varphi), \quad u(0) = \delta, \quad (40)$$

where ξ is a space white noise and δ is the Dirac delta. The existence and uniqueness of u were first established in [HL15] by using a “partial Cole-Hopf transformation” which turns the equation into a well-posed PDE. Using the continuous versions of the objects defined in Sections 3 and 4 we can modify the arguments of [GIP15] to give an alternative proof of their result, see Corollary 5.9 below. The limit of (39) only sees $F'(0)$ and forgets the structure of the non-linearity F , so in that sense the linear parabolic Anderson model arises as a universal scaling limit.

Let us illustrate our result with a (far too simple) model: Suppose F is of the form $F(u) = u(1 - u)$ and let us first consider

$$\partial_t u = \xi \cdot F(u), \quad u(0) \in (0, 1),$$

for some $\xi \in \mathbb{R}$. If $\xi > 0$, then u describes the evolution of the concentration of a growing population in a pleasant environment, which however shows some saturation effects represented by the factor $(1 - u)$. For $\xi < 0$ the individuals live in unfavorable conditions, say in competition with a rival species. From this perspective equation (38) describes the dynamics of a population that migrates between diverse habitats. The meaning of our universality result is that if we tune down the random potential η^ε and counterbalance the growth of the population with some renormalization (think of a death rate), then from far away we can still observe its growth (or extinction) without feeling any saturation effects.

The analysis of (39) and the convergence proof are based on the lattice version of para-controlled distributions that we developed in the previous sections and will be given in Subsection 5.2 below. In that analysis it will be important to understand the limit of $\mathcal{E}^\varepsilon \xi^\varepsilon$ and a certain bilinear functional built from it, and we will also need uniform bounds in suitable Besov spaces. In the following subsection we discuss this convergence.

5.1 Discrete Wick calculus and convergence of the enhanced noise

Here we develop here a general machinery for the use of discrete Wick contractions in the renormalization of discrete, singular SPDEs with i.i.d. noise which is completely analogous to the continuous Gaussian setting. Moreover, we build on the techniques of [CSZ17] to provide a criterion that identifies the scaling limits of discrete Wick products as multiple Wiener-Itô integrals. Our results are summarized in Lemma 5.1 and Lemma 5.4 below and although the use of these results is illustrated only on the discrete parabolic Anderson model, the approach extends in principle to any discrete formulation of popular singular SPDEs such as the KPZ equation or the Φ_d^4 models.

Let us fix a symmetric $\chi \in \mathcal{D}_\omega(\mathbb{R}^d)$, independent of ε , which is 0 on $\frac{1}{4} \cdot \widehat{\mathcal{G}}$ and 1 outside of $\frac{1}{2} \cdot \widehat{\mathcal{G}}$ and define

$$X^\varepsilon := \frac{1}{l^\varepsilon(D)} \chi(D) \xi^\varepsilon.$$

Note that $\mathcal{L}^\varepsilon X^\varepsilon = -L^\varepsilon X^\varepsilon = \chi(D) \xi^\varepsilon$ so that X^ε is a time independent solution to the heat equation on \mathcal{G}^ε induced by our operator \mathcal{L}^ε . Our first task will be to measure the regularity of the sequences (ξ^ε) , (X^ε) in the discrete Besov spaces introduced in Subsection 2.4. For that purpose we need to estimate moments of sufficiently high order. For discrete multiple stochastic integrals with respect to the variables $(\xi^\varepsilon(z))_{z \in \mathcal{G}^\varepsilon}$, that is for sums $\sum_{z_1, \dots, z_n \in \mathcal{G}^\varepsilon} f(z_1, \dots, z_n) \xi^\varepsilon(z_1) \dots \xi^\varepsilon(z_n)$ with $f(z_1, \dots, z_n) = 0$ whenever $z_i = z_j$ for some $i \neq j$ it was shown in [CGP17, Proposition 4.3] that all moments can be bounded in terms of the ℓ^2 norm of f and the corresponding moments of the $(\xi^\varepsilon(z))_{z \in \mathcal{G}^\varepsilon}$. However, typically we will have to bound such expressions for more general f and in that case we first have to arrange our random variable into a finite sum of discrete multiple stochastic integrals, so that then we can apply [CGP17, Proposition 4.3] for each of them. This arrangement can be done in several ways, here we follow [HS15] and regroup in terms of Wick polynomials.

Given random variables $(Y(j))_{j \in J}$ and $I = (j_1, \dots, j_n) \in J^n$ we set

$$Y^I = Y(j_1) \dots Y(j_n) = \prod_{k=1}^n Y(j_k),$$

as well as Y^\emptyset . According to Definition 3.1 and Proposition 3.4 of [LM16], the *Wick product* $:Y^I:$ can be defined recursively by $:Y^\emptyset: := 1$ and

$$:Y^I: := Y^I - \sum_{\emptyset \neq E \subset I} \mathbb{E}[Y^E] :Y^{I \setminus E}:. \quad (41)$$

Lemma 5.1 (see also Proposition 4.3 in [CGP17]). *For $f \in L^2((\mathcal{G}^\varepsilon)^n)$ let*

$$\mathcal{J}_n f := \sum_{z_1, \dots, z_n \in \mathcal{G}^\varepsilon} |\mathcal{G}^\varepsilon|^n f(z_1, \dots, z_n) : \xi^\varepsilon(z_1) \dots \xi^\varepsilon(z_n) :.$$

It then holds for $2 \leq p \leq p_\xi/n$

$$\|\mathcal{J}_n f\|_{L^p(\mathbb{P})} \lesssim \|f\|_{L^2((\mathcal{G}^\varepsilon)^n)}.$$

Proof. In the following we silently identify \mathcal{G}^ε with an enumeration by \mathbb{N}_0 so that we can write

$$\mathcal{J}_n f = \sum_{1 \leq r \leq n, a \in A_r^n} \binom{n}{a} \sum_{z_1 < \dots < z_r} |\mathcal{G}^\varepsilon|^n \tilde{f}_a(z_1, \dots, z_r) \cdot : \xi^\varepsilon(z_1)^{a_1} : \dots : \xi^\varepsilon(z_r)^{a_r} :,$$

where $A_r^n := \{a \in \mathbb{N}_0^r \mid \sum_i a_i = n\}$, \tilde{f}_a denotes the symmetrized version of

$$f_a(z_1, \dots, z_r) := f(\overbrace{z_1, \dots, z_1}^{a_1 \times}, \dots, \overbrace{z_r, \dots, z_r}^{a_r \times}) \cdot \mathbf{1}_{z_i \neq z_j \forall i \neq j},$$

and where we used the independence of $\xi^\varepsilon(z_1), \dots, \xi^\varepsilon(z_r)$ to decompose the Wick product. The independence and the zero mean of the Wick products allow us to see this as a sum of nested martingale transforms so that an iterated application of the Burkholder-Davis-Gundy inequality and Minkowski's inequality as in [CGP17, Proposition 4.3] gives the desired estimate

$$\begin{aligned} \|\mathcal{J}_n f\|_{L^p(\mathbb{P})}^2 &\lesssim \sum_{1 \leq r \leq n, a \in A_r^n} \left\| \sum_{z_1 < \dots < z_r} |\mathcal{G}^\varepsilon|^n \cdot \tilde{f}_a(z_1, \dots, z_r) \cdot : \xi^\varepsilon(z_1)^{a_1} : \dots : \xi^\varepsilon(z_r)^{a_r} : \right\|_{L^p(\mathbb{P})}^2 \\ &\lesssim \sum_{1 \leq r \leq n, a \in A_r^n} \sum_{z_1 < \dots < z_r} |\mathcal{G}^\varepsilon|^{2n} \cdot |\tilde{f}_a(z_1, \dots, z_r)|^2 \cdot \prod_{j=1}^r \|\xi^\varepsilon(z_j)^{a_j}\|_{L^p(\mathbb{P})}^2 \\ &\lesssim \sum_{1 \leq r \leq n, a \in A_r^n} \sum_{z_1, \dots, z_r} |\mathcal{G}^\varepsilon|^n |\tilde{f}_a(z_1, \dots, z_r)|^2 \leq \|f\|_{L^2((\mathcal{G}^\varepsilon)^n)}^2, \end{aligned}$$

where we used the bound $\|\xi^\varepsilon(z_r)^{a_j}\|_{L^p(\mathbb{P})}^2 \lesssim |\mathcal{G}^\varepsilon|^{-a_j}$ which follows from (41). \square

As a direct application we can bound the moments of ξ^ε and X^ε in Besov spaces. Although we will only use the case $d = 2$, here we allow the base space to be a d -dimensional Bravais lattice and define ξ^ε and X^ε analogously in that case. We also need to control the resonant product $X^\varepsilon \circ \xi^\varepsilon$, for which we introduce the renormalization constant

$$c^\varepsilon := \int_{\widehat{\mathcal{G}}^\varepsilon} \frac{\chi(x)}{l^\varepsilon(x)} dx,$$

which is finite for all $\varepsilon > 0$ because $\widehat{\mathcal{G}}^\varepsilon$ is compact and χ is supported away from 0, and we set

$$X^\varepsilon \diamond \xi^\varepsilon := X^\varepsilon \circ \xi^\varepsilon - c^\varepsilon.$$

Remark 5.2. Since $l^\varepsilon \approx |\cdot|^2$ (Lemma 3.5 together with the easy estimate $l^\varepsilon \lesssim |\cdot|^2$) we have $c^\varepsilon \approx -\log \varepsilon$ in dimension 2.

Lemma 5.3. For $\zeta < 2 - d/2 - d/p_\xi$ and $\kappa > d/p_\xi$ we have

$$\mathbb{E} \left[\|\xi^\varepsilon\|_{\mathcal{C}_{\infty}^{\zeta-2}(\mathcal{G}^\varepsilon, p^\kappa)}^{p_\xi} \right] + \mathbb{E} \left[\|X^\varepsilon\|_{\mathcal{C}_{\infty}^{\zeta}(\mathcal{G}^\varepsilon, p^\kappa)}^{p_\xi} \right] + \mathbb{E} \left[\|X^\varepsilon \diamond \xi^\varepsilon\|_{\mathcal{C}_{\infty}^{2\zeta-2}(\mathbb{R}^d, p^{2\kappa})}^{p_\xi/2} \right] \lesssim 1. \quad (42)$$

Proof. Let us bound the regularity of X^ε first. Recall that by Lemma 2.17 we have the continuous embedding (with norm uniformly bounded in ε) $\mathcal{B}_{p_\xi, p_\xi}^{\zeta+d/p_\xi}(\mathcal{G}^\varepsilon, p^\kappa) \subseteq \mathcal{C}_{\infty}^{\zeta}(\mathcal{G}^\varepsilon, p^\kappa)$. To show (42) it is therefore sufficient to bound for $\beta < 2 - d/2$

$$\mathbb{E} \left[\|X^\varepsilon\|_{\mathcal{B}_{p_\xi, p_\xi}^{\beta}(\mathcal{G}^\varepsilon, p^\kappa)}^{p_\xi} \right] = \sum_{-1 \leq j \leq j_{\mathcal{G}^\varepsilon}} 2^{jp_\xi \beta} \sum_{z \in \mathcal{G}^\varepsilon} |\mathcal{G}^\varepsilon| \mathbb{E} [|\Delta_j^{\mathcal{G}^\varepsilon} X^\varepsilon(z)|^{p_\xi}] \frac{1}{(1 + |z|)^{\kappa p_\xi}}.$$

By assumption we have $\kappa p_\xi > d$ and therefore $\sum_{z \in \mathcal{G}^\varepsilon} |\mathcal{G}^\varepsilon| (1 + |z|)^{-\kappa p_\xi}$ is uniformly bounded in ε . It thus suffices to derive a uniform bound for $\mathbb{E}[|\Delta_j^{\mathcal{G}^\varepsilon} X^\varepsilon(x)|^{p_\varepsilon}]$ in ε and x . Note that by (7) $\Delta_j^{\mathcal{G}^\varepsilon} X^\varepsilon(x) = \sum_{z \in \mathcal{G}^\varepsilon} |\mathcal{G}^\varepsilon| \mathcal{K}_j^\varepsilon(x - z) \xi^\varepsilon(z)$ with $\hat{\mathcal{K}}_j^\varepsilon = \varphi_j^{\mathcal{G}^\varepsilon} \chi / l^\varepsilon$ so that Lemma 5.1, Parseval's identity (6) and $l^\varepsilon \gtrsim |\cdot|^2$ imply

$$\mathbb{E}[|\Delta_j X^\varepsilon(x)|^{p_\varepsilon}] \lesssim \|\mathcal{K}_j^\varepsilon\|_{L^2(\mathcal{G}^\varepsilon)}^{p_\varepsilon} \lesssim 2^{jp_\varepsilon(d-2)},$$

which proves the bound for X^ε . The bound for ξ^ε follows from the same arguments or with Lemma 3.4.

Now let us get to $X^\varepsilon \diamond \xi^\varepsilon$. A short computation shows that

$$\mathbb{E}[(X^\varepsilon \diamond \xi^\varepsilon)(x)] = \mathbb{E}[(X^\varepsilon \xi^\varepsilon)(x)] = c^\varepsilon, \quad x \in \mathcal{G}^\varepsilon,$$

and as above it suffices to bound $X^\varepsilon \diamond \xi^\varepsilon$ in $\mathcal{B}_{p_\xi/2, p_\xi/2}^\beta(\mathbb{R}^d, p^{2\kappa})$ for $\beta < 2 - d$. We are therefore left with the task of bounding the $p_\xi/2$ -th moment of $\sum_{|i-j| \leq 1} \Delta_i X^\varepsilon \Delta_j \xi^\varepsilon - \mathbb{E}[\Delta_i X^\varepsilon \Delta_j \xi^\varepsilon]$. But

$$\begin{aligned} & \Delta_i X^\varepsilon(x) \Delta_j \xi^\varepsilon(x) - \mathbb{E}[\Delta_i X^\varepsilon(x) \Delta_j \xi^\varepsilon(x)] \\ &= \sum_{z_1, z_2} |\mathcal{G}^\varepsilon|^2 \mathcal{K}_i^\varepsilon(x - z_1) K_j(x - z_2) (\xi^\varepsilon(z_1) \xi^\varepsilon(z_2) - \mathbb{E}[\xi^\varepsilon(z_1) \xi^\varepsilon(z_2)]) \\ &= \sum_{z_1, z_2} |\mathcal{G}^\varepsilon|^2 \mathcal{K}_i^\varepsilon(x - z_1) K_j(x - z_2) : \xi^\varepsilon(z_1) \xi^\varepsilon(z_2) :, \end{aligned}$$

so that Lemma 5.1 yields

$$\begin{aligned} \mathbb{E} \left[|\Delta_i X^\varepsilon \Delta_j \xi^\varepsilon - \mathbb{E}[\Delta_i X^\varepsilon \Delta_j \xi^\varepsilon]|^{p_\xi/2} \right] &\lesssim \|\mathcal{K}_i^\varepsilon\|_{L^2(\mathcal{G}^\varepsilon)}^{p_\xi/2} \|K_j\|_{L^2(\mathcal{G}^\varepsilon)}^{p_\xi/2} \\ &\lesssim 2^{i(d-4)p_\xi/2} 2^{jd p_\xi/2} \simeq 2^{j(d-2)p_\xi}, \end{aligned}$$

where we used Parseval's identity, $l^\varepsilon \gtrsim |\cdot|^2$ on $\widehat{\mathcal{G}^\varepsilon}$, and that $|i - j| \leq 1$. \square

By the compact embedding result in Lemma 2.18 we see that the sequences $(\mathcal{E}^\varepsilon \xi^\varepsilon)$, $(\mathcal{E}^\varepsilon X^\varepsilon)$, and $(\mathcal{E}^\varepsilon(X^\varepsilon \diamond \xi^\varepsilon))$ have convergent subsequences in distribution. We will see in Lemma 5.5 below that $\mathcal{E}^\varepsilon \xi^\varepsilon$ converges to the white noise ξ on \mathbb{R}^2 . Consequently, the solution X^ε to $-L^\varepsilon X^\varepsilon = \chi(D) \xi^\varepsilon$ converges to the solution of $-LX = \chi(D)\xi$, i.e.

$$X = \frac{1}{(2\pi)^2 \|D\|_\mu^2} \chi(D) \xi = \mathcal{K}^0 * \xi, \quad \mathcal{K}^0 := \overline{\frac{\chi}{(2\pi)^2 \|\cdot\|_\mu^2}}. \quad (43)$$

The limit of $\mathcal{E}^\varepsilon(X^\varepsilon \diamond \xi^\varepsilon)$ will turn out to be the distribution

$$X \diamond \xi(\varphi) := \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \mathcal{K}^0(z_1 - z_2) \varphi(z_1) \xi(dz_1) \xi(dz_2) - (X \prec \xi + \xi \prec X)(\varphi) \quad (44)$$

for $\varphi \in \mathcal{S}_\omega(\mathbb{R}^d)$, where the right hand side denote the second order Wiener-Itô integral with respect to the Gaussian stochastic measure $\xi(dz)$ induced by the white noise ξ , compare [Jan97, Section 7.2]. Note that $X \diamond \xi$ is not a continuous functional of ξ , so the last convergence is not a trivial consequence of the convergence for $\mathcal{E}^\varepsilon \xi^\varepsilon$. To identify the limit of $\mathcal{E}^\varepsilon(X^\varepsilon \diamond \xi^\varepsilon)$ we could use a diagonal sequence argument that first approximates the bilinear functional by a continuous bilinear functional as in [MW17, HS15, CGP17]. Here prefer to go another route and instead we follow [CSZ17] who provide a general criterion for the convergence of discrete multiple stochastic integrals to multiple Wiener-Itô integrals, and we adapt their results to the Wick product setting of Lemma 5.1.

Lemma 5.4 (see also [CSZ17], Theorem 2.3). *Let $n \in \mathbb{N}$ and for $k = 0, \dots, n$ let $f_k^\varepsilon \in L^2((\mathcal{G}^\varepsilon)^k)$. We identify $(\mathcal{G}^\varepsilon)^k$ with a Bravais lattice in kd dimensions via the orthogonal sum $(\mathcal{G}^\varepsilon)^k = \bigoplus_{i=1}^k \mathcal{G}^\varepsilon \subseteq \bigoplus_{i=1}^k \mathbb{R}^d = (\mathbb{R}^d)^k$ to define the Fourier transform $\widehat{f_k^\varepsilon} \in L^2((\widehat{\mathcal{G}^\varepsilon})^k)$ of f_k^ε . Assume that there exist $g_k \in L^2((\mathbb{R}^d)^k)$ with $|\widehat{f_k^\varepsilon} \mathbf{1}_{(\widehat{\mathcal{G}^\varepsilon})^k}| \leq g_k$ for all $\varepsilon \in (0, 1]$ and $\widehat{f_k} \in L^2((\mathbb{R}^d)^k)$ such that $\lim_{\varepsilon \rightarrow 0} \|\widehat{f_k^\varepsilon} \mathbf{1}_{(\widehat{\mathcal{G}^\varepsilon})^k} - \widehat{f_k}\|_{L^2((\mathbb{R}^d)^k)} = 0$ for all $k \leq n$. Then the following convergence holds in distribution*

$$\sum_{k=0}^n \mathcal{J}_k f_k^\varepsilon \longrightarrow \sum_{k=0}^n \int_{(\mathbb{R}^d)^k} f_k(z_1, \dots, z_k) \xi(dz_1) \dots \xi(dz_k),$$

where $f_k \in L^2((\mathbb{R}^d)^k)$ is the inverse Fourier transform of $\widehat{f_k}$.

Proof. The proof is contained in the appendix. \square

The identification of the limits of the extensions of ξ^ε , X^ε and $X^\varepsilon \diamond \xi^\varepsilon$ is a simple application of Lemma 5.4.

Lemma 5.5. *With ξ, X and $\xi \diamond X$ defined as above and with ζ, κ as in Lemma 5.3 we have for $d < 4$*

$$(\mathcal{E}^\varepsilon \xi^\varepsilon, \mathcal{E}^\varepsilon X^\varepsilon, \mathcal{E}^\varepsilon (X^\varepsilon \diamond \xi^\varepsilon)) \xrightarrow{\varepsilon \rightarrow 0} (\xi, X, X \diamond \xi),$$

in distribution in $\mathcal{C}_\infty^{\zeta-2}(\mathbb{R}^d, p^\kappa) \times \mathcal{C}_\infty^\zeta(\mathbb{R}^d, p^\kappa) \times \mathcal{C}_\infty^{2\zeta-2}(\mathbb{R}^d, p^{2\kappa})$.

Proof. Since from Lemma 5.3 we already know that the sequence is tight in $\mathcal{C}_\infty^{\zeta-2}(\mathbb{R}^d, p^\kappa) \times \mathcal{C}_\infty^\zeta(\mathbb{R}^d, p^\kappa) \times \mathcal{C}_\infty^{2\zeta-2}(\mathbb{R}^d, p^{2\kappa})$, it suffices to prove the convergence after testing against $\varphi \in \mathcal{S}_\omega(\mathbb{R}^d)$:

$$(\mathcal{E}^\varepsilon \xi^\varepsilon(\varphi), \mathcal{E}^\varepsilon X^\varepsilon(\varphi), \mathcal{E}^\varepsilon (X^\varepsilon \diamond \xi^\varepsilon)(\varphi)) \xrightarrow{d} (\xi(\varphi), X(\varphi), X \diamond \xi(\varphi)). \quad (45)$$

We can even restrict ourselves to those $\varphi \in \mathcal{S}_\omega(\mathbb{R}^d)$ with $\mathcal{F}_{\mathbb{R}^d} \varphi \in \mathcal{D}_\omega(\mathbb{R}^d)$, which implies $\mathcal{F}_{\mathbb{R}^d}^{-1}(\psi(\varepsilon \cdot) \mathcal{F}_{\mathbb{R}^d} \varphi) = \varphi$ for ε small enough, which we will assume from now on. Let us first show the convergence of (45) in every component.

The convergence of $\mathcal{E}^\varepsilon \xi^\varepsilon$ to the white noise follows from the representation

$$\mathcal{E}^\varepsilon \xi^\varepsilon(\varphi) = \sum_{z \in \mathcal{G}^\varepsilon} |\mathcal{G}^\varepsilon| (\mathcal{F}_{\mathbb{R}^d}^{-1} \psi(\varepsilon \cdot) \mathcal{F}_{\mathbb{R}^d} \varphi)(z) \xi^\varepsilon(z) = \sum_{z \in \mathcal{G}^\varepsilon} |\mathcal{G}^\varepsilon| \varphi(z) \xi^\varepsilon(z)$$

and Lemma 5.4. For the limit of $\mathcal{E}^\varepsilon X^\varepsilon$ we use the formula

$$\mathcal{E}^\varepsilon X^\varepsilon(\varphi) = \sum_{z_1, z_2 \in \mathcal{G}^\varepsilon} |\mathcal{G}^\varepsilon|^2 \varphi(z_1) \mathcal{K}^\varepsilon(z_1 - z_2) \xi^\varepsilon(z_2) = \sum_{z_1, z_2 \in \mathcal{G}^\varepsilon} |\mathcal{G}^\varepsilon|^2 \varphi(z_1) \mathcal{K}^\varepsilon(z_2 - z_1) \xi^\varepsilon(z_2),$$

so that in view of Lemma 5.4 it suffices to note that $\widehat{f}^\varepsilon(x) := \widehat{\varphi}(x) \chi(x)/l^\varepsilon(x)$ is dominated by $\chi/|\cdot|^2$ and converges to $\chi/((2\pi)^2 \|\cdot\|_\mu^2)$ on $\widehat{\mathcal{G}^\varepsilon}$.

We are left with the convergence of the third component. Since $\mathcal{E}^\varepsilon \xi^\varepsilon \rightarrow \xi$ and $\mathcal{E}^\varepsilon X^\varepsilon \rightarrow X$, which implies

$$\lim_{\varepsilon \rightarrow 0} \mathcal{E}^\varepsilon (X^\varepsilon \diamond \xi^\varepsilon) = \lim_{\varepsilon \rightarrow 0} \mathcal{E}^\varepsilon X^\varepsilon \diamond \mathcal{E}^\varepsilon \xi^\varepsilon = X \diamond \xi$$

and similarly $\mathcal{E}^\varepsilon(\xi^\varepsilon \prec X^\varepsilon) \rightarrow \xi \prec X$, we can instead show

$$\mathcal{E}^\varepsilon(X^\varepsilon \xi^\varepsilon - \mathbb{E}[X^\varepsilon \xi^\varepsilon])(\varphi) \rightarrow (X \diamond \xi + \xi \prec X + X \prec \xi)(\varphi). \quad (46)$$

Note that we have the representations

$$\begin{aligned} \mathcal{E}^\varepsilon(X^\varepsilon \xi^\varepsilon - \mathbb{E}[X^\varepsilon \xi^\varepsilon])(\varphi) &= \sum_{z_1, z_2 \in \mathcal{G}^\varepsilon} |\mathcal{G}^\varepsilon|^2 \varphi(z_1) \mathcal{K}^\varepsilon(z_1 - z_2) : \xi^\varepsilon(z_1) \xi^\varepsilon(z_2) :, \\ (X \diamond \xi + \xi \prec X + X \prec \xi)(\varphi) &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \varphi(z_1) \mathcal{K}^0(z_1 - z_2) \xi(dz_1) \xi(dz_2). \end{aligned}$$

The $(\mathcal{G}^\varepsilon)^2$ -Fourier transform of $\varphi(z_1) \mathcal{K}^\varepsilon(z_1 - z_2)$ is $\hat{\varphi}_{\text{per}}(x_1 - x_2) \chi(x_2)/l^\varepsilon(x_2)$ for $x_1, x_2 \in \widehat{\mathcal{G}^\varepsilon}$, where $\hat{\varphi}_{\text{per}}$ denotes the \mathcal{R} -periodic extension of $\hat{\varphi} \in \mathcal{D}_\omega(\mathbb{R}^d)$. We can therefore apply Lemma 5.4 since for $d < 4$ $(\chi(x_2)/l^\varepsilon(x_2)) \lesssim \mathbf{1}_{|x| \gtrsim 1}/|x|^4$ is integrable on $\widehat{\mathcal{G}^\varepsilon}$ and thus we obtain (46).

We have shown the convergence in distribution of all the components in (45). By Lemma 5.4 we can take any linear combination of these components and still get the convergence from the same estimates, so (45) follows from the Cramér-Wold Theorem. \square

5.2 Convergence of the lattice model

We are now ready to prove the convergence of $\mathcal{E}^\varepsilon u^\varepsilon$ announced at the beginning of this section. The key statement will be the a priori estimate in Lemma 5.7. The convergence of $\mathcal{E}^\varepsilon u^\varepsilon$ to the continuous solution on \mathbb{R}^2 , constructed in Corollary 5.9, will be proven in Theorem 5.10. We first fix the relevant parameters.

Preliminaries

Throughout this subsection we use the same $p \in [1, \infty]$, $\sigma \in (0, 1)$, a polynomial weight p^κ for some $\kappa > 2/p_\xi > 1/7$ and a time dependent sub-exponential weight $(e_{l+t}^\sigma)_{t \in [0, T]}$. We further fix an arbitrarily large time horizon $T > 0$ and require $l \leq -T$ for the parameter in the weight e_l^σ . Then we have $1 \leq e_{l+t}^\sigma \leq (e_{l+t}^\sigma)^2$ for any $t \leq T$, which will be used to control a quadratic term that comes from the Taylor expansion of the non-linearity F^ε .

In this subsection we fix a parameter

$$\alpha \in (2/3 - 2/3 \cdot \kappa/\sigma, 1 - 2/p_\xi - 2\kappa/\sigma) \quad (47)$$

with $\kappa/\sigma \in (2/p_\xi, 1)$ small enough such that the interval in is non-empty, which is possible since $2/p_\xi < 1/7$. Let us mention the simple facts $2\alpha + 2\kappa/\sigma, 2\alpha + 4\kappa/\sigma \in (0, 2)$, $\alpha + \kappa/\sigma, \alpha + 2\kappa/\sigma \in (0, 1)$ and $3\alpha + 2\kappa/\sigma - 2 > 0$ which we will use frequently below.

We will assume that the initial conditions u_0^ε are uniformly bounded in $\mathcal{C}_p^0(\mathcal{G}^\varepsilon, e_l^\sigma)$ and such that $\mathcal{E}^\varepsilon u_0^\varepsilon$ converges in $\mathcal{S}'_\omega(\mathbb{R}^2)$ to some u_0 . For $u_0^\varepsilon = |\mathcal{G}^\varepsilon|^{-1} \mathbf{1}_{=0}$ it is easily verified that this is indeed the case and the limit is the Dirac delta, $u_0 = \delta$.

Recall that we aim at showing that (the extension of) the solution u^ε to

$$\mathcal{L}^\varepsilon u^\varepsilon = F(u^\varepsilon)(\xi^\varepsilon - c^\varepsilon), \quad u^\varepsilon(0) = u_0^\varepsilon = |\mathcal{G}^\varepsilon|^{-1} \mathbf{1}_{=0} \quad (48)$$

converges to the solution of

$$\mathcal{L}u = F'(0)u \diamond \xi, \quad u(0) = u_0 = \delta, \quad (49)$$

where $u \diamond \xi$ is a suitably renormalized product defined in Corollary 5.9 below. Our solutions will be objects in the parabolic space $\mathcal{L}_{p,T}^{\alpha,\alpha}$ which does not require continuity at $t = 0$. A priori there is thus no obvious meaning for the Cauchy problems (48), (49) (although of course for (48) we could use the pointwise interpretation). We follow the common interpretation for distributions $u^\varepsilon, u \in \mathcal{D}'_\omega(\mathbb{R}^{1+2})$ (compare for example [Tri92, Definition 3.3.4]) to require $\text{supp } u^\varepsilon, \text{supp } u \subseteq \mathbb{R}_+ \times \mathbb{R}^2$ and

$$\begin{aligned}\mathcal{L}^\varepsilon u^\varepsilon &= F(u^\varepsilon)(\xi^\varepsilon - c^\varepsilon) + \delta \otimes u_0^\varepsilon, \\ \mathcal{L}u &= F'(0)u \diamond \xi + \delta \otimes u_0,\end{aligned}$$

in the distributional sense on $(-\infty, T)$, where \otimes denotes the tensor product between distributions. Since we mostly work with the mild formulation of these equations the distributional interpretation will not play a crucial role. Some care is needed to check that the only distributional solutions are mild solutions, since the distributional Cauchy problem for the heat equation is not uniquely solvable [Tyc35]. However, under generous growth conditions for u, u^ε for $x \rightarrow \infty$ (compare [Fri64]) there is a unique solution. In our case this fact can be checked by considering the Fourier transform of u, u^ε in space.

A priori estimates

We will work with the following space of paracontrolled distributions.

Definition 5.6 (Paracontrolled distribution for 2d PAM). *We identify a pair*

$$(u^{\varepsilon,X}, u^{\varepsilon,\sharp}): [0, T] \rightarrow \mathcal{S}'_\omega(\mathcal{G}^\varepsilon)^2$$

with $u^\varepsilon \in \mathcal{S}'_\omega(\mathcal{G}^\varepsilon)$ via $u^\varepsilon = u^{\varepsilon,X} \ll X^\varepsilon + u^{\varepsilon,\sharp}$ and introduce a norm

$$\|u^\varepsilon\|_{\mathcal{D}_{p,T}^{\gamma,\delta}} := \|(u^{\varepsilon,X}, u^{\varepsilon,\sharp})\|_{\mathcal{D}_{p,T}^{\gamma,\delta}} := \|u^{\varepsilon,X}\|_{\mathcal{L}_{p,T}^{\gamma/2,\delta}(\mathcal{G}^\varepsilon, e_l^\sigma)} + \|u^{\varepsilon,\sharp}\|_{\mathcal{L}_{p,T}^{\delta+\alpha}(\mathcal{G}^\varepsilon, e_l^\sigma)} \quad (50)$$

for α as above and $\gamma \geq 0, \delta > 0$. We denote the corresponding space by $\mathcal{D}^{\gamma,\delta}(\mathcal{G}^\varepsilon, e_l^\sigma)$. If the norm (50) is bounded for a sequence $u^\varepsilon = u^{\varepsilon,X} \ll X^\varepsilon + u^{\varepsilon,\sharp}$ we say that u^ε is paracontrolled by X^ε .

As in [GP15b] it will be useful to have a common bound on the data: let $M > 0$ be such that (compared to Lemma 5.3 we have $\zeta = \alpha + 2\kappa/\sigma$)

$$\|\xi^\varepsilon\|_{\mathcal{C}_\infty^{\alpha+2\kappa/\sigma-2}(\mathcal{G}^\varepsilon, p^\kappa)} \vee \|X^\varepsilon\|_{\mathcal{C}_\infty^{\alpha+2\kappa/\sigma}(\mathcal{G}^\varepsilon, p^\kappa)} \vee \|X^\varepsilon \diamond \xi^\varepsilon\|_{\mathcal{C}_\infty^{2\alpha+4\kappa/\sigma-2}(\mathcal{G}^\varepsilon, p^{2\kappa})} \leq M. \quad (51)$$

The following a priori estimates will allow us to set up a Picard iteration below.

Lemma 5.7 (A priori estimates). *Given $u^\varepsilon = u^{\varepsilon,X} \ll X^\varepsilon + u^{\varepsilon,\sharp}$ define $v^\varepsilon, v^{\varepsilon,\sharp}$ by*

$$\mathcal{L}^\varepsilon v^\varepsilon := F^\varepsilon(u^\varepsilon)\xi^\varepsilon - F^\varepsilon(u^{\varepsilon,X}/F'(0))F'(0)c^\varepsilon, \quad v^\varepsilon(0) = u^\varepsilon(0), \quad (52)$$

$$v^{\varepsilon,\sharp} := v^\varepsilon - F'(0)u^\varepsilon \ll X^\varepsilon. \quad (53)$$

We then have for $\gamma \in \{0, \alpha\}$ the bound

$$\begin{aligned}\|(F'(0)v^\varepsilon, v^{\varepsilon,\sharp})\|_{\mathcal{D}_{p,T}^{\gamma,\alpha}} &\lesssim_M \mathbf{1}_{\gamma=0} \left(\|v^{\varepsilon,\sharp}(0)\|_{\mathcal{C}_p^{2\alpha}(\mathcal{G}^\varepsilon, e_l^\sigma)} + \|u^{\varepsilon,\sharp}(0)\|_{\mathcal{C}_p^{2\alpha}(\mathcal{G}^\varepsilon, e_l^\sigma)} + \|u^{\varepsilon,X}(0)\|_{\mathcal{C}_p^\alpha(\mathcal{G}^\varepsilon, e_l^\sigma)} \right) \\ &\quad + \mathbf{1}_{\gamma=\alpha} \left(\|v^{\varepsilon,\sharp}(0)\|_{\mathcal{C}_p^0(\mathcal{G}^\varepsilon, e_l^\sigma)} \right) + T^{(\alpha-\delta)/2} \left(\|u^\varepsilon\|_{\mathcal{D}_{p,T}^{\gamma,\alpha}} + \varepsilon^\nu \|u^\varepsilon\|_{\mathcal{D}_{p,T}^{\gamma,\alpha}}^2 \right)\end{aligned}$$

for $\delta \in (2 - 2\alpha - 2\kappa/\sigma, \alpha)$, M as in (51) and some $\nu > 0$. The involved constant can be chosen proportional to $(1 + \|F''\|_{L^\infty(\mathbb{R})})(1 + M^2)$.

Remark 5.8. *The complicated formulation of (52) is necessary because when we expand the singular product on the right hand side we get*

$$F^\varepsilon(u^\varepsilon)\xi^\varepsilon = F'(0)(C(u^{\varepsilon,X}, X^\varepsilon, \xi^\varepsilon) + u^{\varepsilon,X}(X^\varepsilon \circ \xi^\varepsilon)) + \dots,$$

so to obtain the right renormalization we need to subtract $F'(0)u^{\varepsilon,X}c^\varepsilon$, which is exactly what we get if we Taylor expand the second addend on the right hand side of (52). Of course, if u is a fixed point of the map defined in (52), (53), then $u^{\varepsilon,X} = F'(0)u^\varepsilon$ and the “renormalization term” is just $F^\varepsilon(u^\varepsilon)F'(0)c^\varepsilon$.

Proof. The solution to (52), (53) can be constructed using the Green’s function $\widetilde{e^{-t\ell^\varepsilon}}$ and Duhamel’s principle. We derive the bounds similar in spirit to [GP15b]. To uncluster the notation a bit, we will drop the upper index ε on $u, v, X, \mathcal{L}, \dots$ in this proof. We show both estimates at once by denoting by γ either 0 or α .

Throughout the proof we will use the fact that

$$\|u\|_{\mathcal{L}_{p,T}^{\gamma/2,\alpha}(\mathcal{G}^\varepsilon, e_l^\sigma)} = \|u^X \ll X + u^\sharp\|_{\mathcal{L}_{p,T}^{\gamma/2,\alpha}(\mathcal{G}^\varepsilon, e_l^\sigma)} \lesssim \|u\|_{\mathcal{D}_{p,T}^{\gamma,\beta}} \quad (54)$$

for $\beta \in (0, \alpha]$ which follows from Lemma 4.6. We first estimate

$$\|v\|_{\mathcal{L}_{p,T}^{\gamma/2,\alpha}(\mathcal{G}^\varepsilon, e_l^\sigma)} \lesssim \|F'(0)u \ll X + v^\sharp\|_{\mathcal{L}_{p,T}^{\gamma/2,\alpha}(\mathcal{G}^\varepsilon, e_l^\sigma)} \stackrel{(54)}{\lesssim} \|u\|_{\mathcal{D}_{p,T}^{\gamma,\delta}} + \|v^\sharp\|_{\mathcal{L}_{p,T}^{\gamma,2\alpha}(\mathcal{G}^\varepsilon, e_l^\sigma)},$$

where we used Lemma 4.5 and Lemma 3.10 in the second step. This leaves us with the task of estimating $\|v^\sharp\|_{\mathcal{L}_{p,T}^{\gamma,2\alpha}(\mathcal{G}^\varepsilon, e_l^\sigma)}$. We split

$$\begin{aligned} \mathcal{L}v^\sharp &= \mathcal{L}(v - F'(0)u \ll X) \\ &= F'(0)u\xi - F^\varepsilon(u^X/F'(0))F'(0)c - F'(0)\mathcal{L}(u \ll X) + R(u)u^2\xi \\ &= F'(0)[u \prec (\xi - \bar{\xi}) + u \prec \bar{\xi} - u \ll \bar{\xi} + u \ll \bar{\xi} - \mathcal{L}(u \ll X) + \xi \prec u] \\ &\quad + C(u^X, X, \xi) + u^X(X \diamond \xi) \\ &\quad + u^\sharp \circ \xi] \\ &\quad + R(u) \cdot u^2\xi \\ &\quad - R(u^X) \cdot (u^X)^2c/F'(0), \end{aligned} \quad \begin{aligned} (<) \\ (\circ) \\ (\sharp) \\ (R_u) \\ (R_{uX}) \end{aligned}$$

where $\bar{\xi} = \chi(D)\xi$ so that $\mathcal{L}X = \bar{\xi}$ and $\xi - \bar{\xi} \in \bigcap_{\beta \in \mathbb{R}} \mathcal{C}_\infty^\beta(\mathcal{G}^\varepsilon, p^\kappa)$ and where $R(u) = \varepsilon^2 \int_0^1 F''(\lambda \varepsilon^2 u) d\lambda$. We have by Lemmas 4.2, 4.7 $\|(<)\|_{\mathcal{M}_T^\gamma \mathcal{C}_p^{2\alpha+2\kappa/\sigma-2}(\mathcal{G}^\varepsilon, e_l^\sigma p^\kappa)} \lesssim \|u\|_{\mathcal{L}_{p,T}^{\gamma/2,\alpha}(\mathcal{G}^\varepsilon, e_l^\sigma)} \stackrel{(54)}{\lesssim} \|u\|_{\mathcal{D}_{p,T}^{\gamma,\delta}}$ and further with Lemma 4.3 and Lemma 4.2 $\|(\circ)\|_{\mathcal{M}_T^\gamma \mathcal{C}^{2\alpha+4\kappa/\sigma-2}(\mathcal{G}^\varepsilon, e_l^\sigma p^{2\kappa})} \lesssim \|u\|_{\mathcal{D}_{p,T}^{\gamma,\delta}}$, while the term (\sharp) can be bounded with Lemma 4.2 by $\|u^\sharp \circ \xi\|_{\mathcal{M}_T^\gamma \mathcal{C}_p^{2\alpha+2\kappa/\sigma-2}(\mathcal{G}^\varepsilon, e_l^\sigma p^\kappa)} \lesssim \|u^\sharp\|_{\mathcal{L}_{p,T}^{\gamma,\alpha+\delta}(\mathcal{G}^\varepsilon, e_l^\sigma)} \leq \|u\|_{\mathcal{D}_{p,T}^{\gamma,\delta}}$. To estimate (R_u) we use the simple bounds $\|\varepsilon^{\beta'} f\|_{\mathcal{C}_q^{\beta+\beta'}(\mathcal{G}^\varepsilon, \rho)} \lesssim \|f\|_{\mathcal{C}_q^\beta(\mathcal{G}^\varepsilon, \rho)}$ for $\beta \in \mathbb{R}$, $\beta' > 0$, $q \in [1, \infty]$, $\rho \in \boldsymbol{\rho}(\omega)$ and $\|\varepsilon^{-\beta} f\|_{L^q(\mathcal{G}^\varepsilon, \rho)} \lesssim \varepsilon^{-\beta} \sum_{j \lesssim j_{\mathcal{G}^\varepsilon}} 2^{-j\beta} \|f\|_{\mathcal{C}_q^\beta(\mathcal{G}^\varepsilon, \rho)} \lesssim \|f\|_{\mathcal{C}_q^\beta(\mathcal{G}^\varepsilon, \rho)}$ for

$\beta < 0$, $q \in [1, \infty]$, $\rho \in \boldsymbol{\rho}(\omega)$ and the assumption $F'' \in L^\infty$ and obtain for $\nu' > 0$

$$\begin{aligned}
\|(R_u)\|_{\mathcal{M}_T^\gamma \mathcal{C}_p^{2\alpha+2\kappa/\sigma-2}(\mathcal{G}^\varepsilon, e_l^\sigma p^\kappa)} &\lesssim \|F''\|_\infty \|\varepsilon^{\alpha+2\kappa/\sigma} u^2\|_{\mathcal{M}^\gamma L^p(\mathcal{G}^\varepsilon, e_l^\sigma)} \|\varepsilon^{2-(\alpha+2\kappa/\sigma)} \xi\|_{L^\infty(\mathcal{G}^\varepsilon, p^\kappa)} \\
&\lesssim \|\varepsilon^{\alpha+2\kappa/\sigma} u^2\|_{\mathcal{M}_T^\gamma L^p(\mathcal{G}^\varepsilon, (e_l^\sigma)^2)} \|\xi\|_{\mathcal{C}_\infty^{\alpha+2\kappa/\sigma-2}(\mathcal{G}^\varepsilon, p^\kappa)} \\
&\lesssim \|\varepsilon^{\alpha/2+\kappa/\sigma} u\|_{\mathcal{M}_T^{\gamma/2} L^{2p}(\mathcal{G}^\varepsilon, e_l^\sigma)}^2 \lesssim \|\varepsilon^{\alpha/2+\kappa/\sigma} u\|_{\mathcal{M}_T^{\gamma/2} \mathcal{C}_p^{d/2p+\nu'}(\mathcal{G}^\varepsilon, e_l^\sigma)}^2 \\
&\leq \|\varepsilon^{\alpha/2+\kappa/\sigma} u\|_{\mathcal{M}_T^{\gamma/2} \mathcal{C}_p^{1+\nu'}(\mathcal{G}^\varepsilon, e_l^\sigma)}^2 \lesssim \|\varepsilon^{\alpha/2+\kappa/\sigma-(1+\nu'-\alpha)} u\|_{\mathcal{M}_T^{\gamma/2} \mathcal{C}_p^\alpha(\mathcal{G}^\varepsilon, e_l^\sigma)}^2 \\
&\lesssim \varepsilon^{3\alpha+2\kappa/\sigma-2(1+\nu')} \|u\|_{\mathcal{D}_{p,T}^{\gamma,\delta}}^2,
\end{aligned}$$

so that for sufficiently small $\nu' > 0$ we can choose $\nu \in (0, 3\alpha + 2\kappa/\sigma - 2(1 + \nu'))$. Similarly we get for (a different) $\nu' \in (0, \delta)$

$$\begin{aligned}
\|(R_{u^X})\|_{\mathcal{M}_T^\gamma \mathcal{C}_p^{2\alpha+2\kappa/\sigma-2}(\mathcal{G}^\varepsilon, e_l^\sigma p^\kappa)} &\lesssim \|F''\|_\infty c \|\varepsilon u^X\|_{\mathcal{M}_T^{\gamma/2} L^{2p}(\mathcal{G}^\varepsilon, e_l^\sigma)}^2 \lesssim c \|\varepsilon u^X\|_{\mathcal{M}_T^{\gamma/2} \mathcal{C}_p^{1+\nu'}(\mathcal{G}^\varepsilon, e_l^\sigma)}^2 \\
&\lesssim \varepsilon^{2(\delta-\nu')} \log(\varepsilon) \|u^X\|_{\mathcal{M}_T^{\gamma/2} \mathcal{C}_p^\delta(\mathcal{G}^\varepsilon, e_l^\sigma)}^2 \lesssim \varepsilon^\nu \|u\|_{\mathcal{D}_{p,T}^{\gamma,\delta}}^2.
\end{aligned}$$

where we chose $\nu \in (0, \delta - \nu')$. The Schauder estimates of Lemma 3.9 yield on these grounds

$$\begin{aligned}
\|v^\sharp\|_{\mathcal{D}_{p,T}^{\gamma,2\alpha}(\mathcal{G}^\varepsilon, e_l^\sigma)} &\lesssim \mathbf{1}_{\gamma=\alpha} \|v^\sharp(0)\|_{\mathcal{C}_p^0(\mathcal{G}^\varepsilon, e_l^\sigma)} + \mathbf{1}_{\gamma=0} \|v^\sharp(0)\|_{\mathcal{C}_p^{2\alpha}(\mathcal{G}^\varepsilon, e_l^\sigma)} + \|u\|_{\mathcal{D}_{p,T}^{\gamma,\delta}} + \varepsilon^\nu \|u\|_{\mathcal{D}_{p,T}^{\gamma,\delta}}^2 \\
&\lesssim \mathbf{1}_{\gamma=\alpha} \|v^\sharp(0)\|_{\mathcal{C}_p^0(\mathcal{G}^\varepsilon, e_l^\sigma)} + T^{(\alpha-\delta)/2} (\|u\|_{\mathcal{D}_{p,T}^{\gamma,\alpha}} + \varepsilon^\nu \|u\|_{\mathcal{D}_{p,T}^{\gamma,\alpha}}^2) \\
&\quad + \mathbf{1}_{\gamma=0} \left(\|v^\sharp(0)\|_{\mathcal{C}_p^{2\alpha}(\mathcal{G}^\varepsilon, e_l^\sigma)} + \|u^{\varepsilon,\sharp}(0)\|_{\mathcal{C}_p^{2\alpha}(\mathcal{G}^\varepsilon, e_l^\sigma)} + \|u^{\varepsilon,X}(0)\|_{\mathcal{C}_p^\alpha(\mathcal{G}^\varepsilon, e_l^\sigma)} \right),
\end{aligned}$$

where in the last step we used Lemma 3.10. \square

Convergence to the continuum

It is straightforward to redo our computations in the continuous case which leads to the existence of a solution to the continuous linear parabolic Anderson model on \mathbb{R}^2 , a result which was already established in [HL15]. Since the continuous analogue of our approach is a one-to-one translation of the discrete statements and definitions above we do not provide the details.

Corollary 5.9. *For any $u_0 \in \mathcal{C}_p^0(\mathbb{R}^d, e_l^\sigma)$ there is a unique solution $u = F'(0)u \llcorner X + u^\sharp \in \mathcal{D}_{p,T}^{\gamma,\beta}(\mathbb{R}^d, e_l^\sigma)$, $\beta \in (2/3, 1)$, $\gamma \in [\beta, 1)$ to*

$$\mathcal{L}u = F'(0)u \diamond \xi, \quad u(0) = u_0,$$

where ξ is white noise on \mathbb{R}^2 , \mathcal{L} is defined as in section 3 and where

$$u \diamond \xi := \xi \llcorner u + u \llcorner \xi + F'(0)C(u, X, \xi) + F'(0)u(X \diamond \xi) + u^\sharp \circ \xi$$

with X , $X \diamond \xi$ as in (43), (44).

Sketch of the proof. Redoing the computations in the continuous case leads to the continuous version of the a priori estimates of Lemma 5.7, without the quadratic term:

$$\begin{aligned}
\|(F'(0)v, v^\sharp)\|_{\mathcal{D}_{p,T}^{\gamma,\beta}} &\lesssim_M \|v^\sharp(0)\|_{\mathcal{C}_p^0(\mathbb{R}^d, e_l^\sigma)} + T^{(\beta-\delta)/2} \|u\|_{\mathcal{D}_{p,T}^{\gamma,\beta}} \\
\|(F'(0)v, v^\sharp)\|_{\mathcal{D}_{p,T}^{0,\beta}} &\lesssim_M \|v^\sharp(0)\|_{\mathcal{C}_p^{2\beta}(\mathbb{R}^d, e_l^\sigma)} + \|u^\sharp(0)\|_{\mathcal{C}_p^{2\beta}(\mathbb{R}^d, e_l^\sigma)} + \|u^X(0)\|_{\mathcal{C}_p^\beta(\mathbb{R}^d, e_l^\sigma)} + T^{(\beta-\delta)/2} \|u\|_{\mathcal{D}_{p,T}^{0,\beta}}
\end{aligned}$$

for $v = F'(0)u \ll X + v^\sharp$, $\mathcal{L}v = F'(0)u \diamond \xi$, $v(0) = u(0) = u_0$. Choosing $T > 0$ small enough we can set up a Picard iteration (e.g. starting in $t \mapsto e^{t\mathcal{L}}u_0 =: 0 \ll X + u^\sharp$) where we use either the first or the second estimate depending on the smoothness of the initial condition and obtain a bounded sequence in $\mathcal{D}_{p,T}^{\gamma,\beta}(\mathbb{R}^d, e_l^\sigma)$. The limit of this iteration (maybe after passing to a subsequence) is a local solution u , and as in [GP15b, Theorem 6.12]) those local solutions can be concatenated to a paracontrolled solution $u = F'(0)u \ll X + u^\sharp \in \mathcal{D}_{p,T}^{\gamma,\beta}(\mathbb{R}^d, e_l^\sigma)$ on $[0, T]$.

To verify uniqueness one can use that two different solutions $u = F'(0)u \ll X + u^\sharp$, $v = F'(0)v \ll X + v^\sharp$ for the same initial data have a difference $u - v = (u - v) \ll X + (u^\sharp - v^\sharp)$ that solves once more the linear parabolic Anderson model with initial condition 0 so that the a priori estimates above give $u - v = 0$. \square

We can now deduce the main theorem of this section, where the parameters are as defined above.

Theorem 5.10. *Let u_0^ε be a uniformly bounded sequence in $\mathcal{C}_p^0(\mathcal{G}^\varepsilon, e_l^\sigma)$ such that $\mathcal{E}^\varepsilon u_0^\varepsilon$ converges to some u_0 in $\mathcal{S}'_\omega(\mathbb{R}^2)$. Then there are unique solutions $u^\varepsilon \in \mathcal{D}_{p,T^\varepsilon}^{\alpha,\alpha'}(\mathcal{G}^\varepsilon, e_l^\sigma)$ to*

$$\mathcal{L}^\varepsilon u^\varepsilon = F^\varepsilon(u^\varepsilon)(\xi^\varepsilon - c^\varepsilon F'(0)), \quad u^\varepsilon(0) = u_0^\varepsilon,$$

on $[0, T^\varepsilon)$ with $T^\varepsilon := T \wedge \sup_{t \geq 0} \{ \|u^\varepsilon(t)\|_{\mathcal{D}_{p,T}^{\alpha,\alpha}} < \infty \}$. It holds $T^\varepsilon = T$ for ε small enough. The sequence $u^\varepsilon = F'(0)u^\varepsilon \ll X + u^{\varepsilon,\sharp} \in \mathcal{D}_{p,T}^{\alpha,\alpha'}(\mathcal{G}^\varepsilon, e_l^\sigma)$ is uniformly bounded (for ε small enough such that $T = T^\varepsilon$). Their extensions $\mathcal{E}^\varepsilon u^\varepsilon$ converge in distribution in $\mathcal{D}_{p,T}^{\alpha,\alpha'}(\mathbb{R}^d, e_l^{\sigma'})$, $\alpha' < \alpha$, $\sigma' < \sigma$, to the solution u of the linear equation in Corollary 5.9.

Remark 5.11. *Since T^ε is a random time the convergence in distribution has to be defined with some care: We say that $u^\varepsilon \rightarrow u$ in distribution if for any $f \in C_b(\mathcal{D}_{p,T}^{\alpha,\alpha'}(\mathcal{G}^\varepsilon, e_l^\sigma); \mathbb{R})$, which we extend to exploding paths by simply setting it to 0, we have $\mathbb{E}[f(u^\varepsilon)] = \mathbb{E}[f(u^\varepsilon)\mathbf{1}_{T^\varepsilon \leq T}] \rightarrow \mathbb{E}[f(u)]$ and further $\mathbb{P}(T^\varepsilon \leq T) \rightarrow 0$.*

Proof. Existence of and uniform bounds for a solution u^ε follow similarly as in Corollary 5.9 with the only difference that, due to the presence of the quadratic term in the a priori estimates, the time T_*^ε on which a Picard iteration can be set up is now of the form

$$T_*^\varepsilon = T_1 \varepsilon^{-\frac{2\nu}{\alpha-\delta}} \wedge T_2$$

with $T \geq T_2 > 0$ independent of ε and $T_1 > 0$ depending on the sequence of initial conditions (but independent of ε). Therefore, we can concatenate the paracontrolled solutions up to the blow-up time T^ε , which by the shape of T_*^ε coincides with T for ε small enough.

To check the uniqueness of the discrete equation suppose that we are given two solutions $u^\varepsilon, v^\varepsilon$, which then satisfy

$$\begin{aligned} \mathcal{L}^\varepsilon(u^\varepsilon - v^\varepsilon) &= (F^\varepsilon(u^\varepsilon) - F^\varepsilon(v^\varepsilon))(\xi^\varepsilon - c^\varepsilon F'(0)) \\ &= \underbrace{\int_0^1 F'(u^\varepsilon + \zeta(v^\varepsilon - u^\varepsilon)) d\zeta}_{=: \mathcal{F}} \cdot (v^\varepsilon - u^\varepsilon)(\xi^\varepsilon - c^\varepsilon F'(0)). \end{aligned}$$

We already know, by the a priori estimates, that $u^\varepsilon = F'(0)u^\varepsilon \ll X^\varepsilon + u^{\varepsilon,\sharp}$, $v^\varepsilon = F'(0)v^\varepsilon \ll X^\varepsilon + v^{\varepsilon,\sharp}$ are bounded in $\mathcal{D}_{p,T_*^\varepsilon}^{\alpha,\alpha'}(\mathcal{G}^\varepsilon, e_l^\sigma)$. As we only care now to prove uniqueness for a fixed scale ε we

do not care about picking up negative powers of ε so that we can consider our equation started in “paracontrolled” initial conditions $u^\varepsilon(0) = v^\varepsilon(0) \in \mathcal{C}_p^\alpha(\mathcal{G}^\varepsilon, e_l^\sigma)$, $u^{\varepsilon,\sharp}(0) = v^{\varepsilon,\sharp}(0) \in \mathcal{C}_p^{2\alpha}(\mathcal{G}^\varepsilon, e_l^\sigma)$ and our solutions contained in $\mathcal{D}_{p,T_*^\varepsilon}^{0,\alpha}(\mathcal{G}^\varepsilon, e_l^\sigma)$. Consequently, since e_l^σ is an increasing function, the integral term \mathcal{F} is an object in $L^\infty(\mathcal{G}^\varepsilon)$ and by picking up a further negative power of ε we can consider it as an element of $\mathcal{M}_{T_*^\varepsilon}^0 \mathcal{C}_\infty^\beta(\mathcal{G}^\varepsilon)$ for any $\beta \in \mathbb{R}$. The product $(v^\varepsilon - u^\varepsilon)(\xi^\varepsilon - c^\varepsilon F'(0))$ can be estimated as in the proof of Lemma 5.7. Since multiplication by \mathcal{F} only contributes an $(\varepsilon$ -dependent) factor we obtain a bound of the form

$$\|u^\varepsilon - v^\varepsilon\|_{\mathcal{D}_{p,T_*^\varepsilon}^{0,\alpha}} \lesssim_\varepsilon (T_*^\varepsilon)^{\frac{\alpha-\delta}{2}} \|u^\varepsilon - v^\varepsilon\|_{\mathcal{D}_{p,T_*^\varepsilon}^{0,\alpha}},$$

which shows $\|u^\varepsilon - v^\varepsilon\|_{\mathcal{D}_{p,T}^{0,\alpha}} = 0$ for T_*^ε small enough. Iterating this argument gives $u^\varepsilon = v^\varepsilon$ on all of $[0, T^\varepsilon)$.

It remains to show that this unique solution $\mathcal{E}u^\varepsilon$ converges to u . By Skohorod representation we know that $\mathcal{E}^\varepsilon \xi^\varepsilon$, $\mathcal{E}^\varepsilon X^\varepsilon$, $\mathcal{E}^\varepsilon(X^\varepsilon \diamond \xi^\varepsilon)$ in Lemma 5.5 converge almost surely on a suitable probability space. We will work on this space from now on. The application of the Skohorod representation theorem is indeed allowed since the limiting measure of these objects has support in the closure of smooth functions and thus in a separable space. Having proved that the sequence u^ε is uniformly bounded in $\mathcal{D}_{p,T^\varepsilon}^{\alpha,\alpha}(\mathcal{G}^\varepsilon, e_l^\sigma)$ we know that $\mathcal{E}^\varepsilon u^\varepsilon$ is uniformly bounded in $\mathcal{D}_{p,T}^{\alpha,\alpha}(\mathbb{R}^d, e_l^\sigma)$ (for $\varepsilon > 0$ small enough such that $T^\varepsilon = T$). To show the convergence we note that by we can apply compact embedding arguments and obtain a convergent subsequence of $\mathcal{E}^\varepsilon u^\varepsilon$ that converges to some $u = F'(0)u \ll X + u^\sharp \in \mathcal{D}_{p,T}^{\alpha,\alpha'}(\mathbb{R}^d, e_l^{\sigma'})$ in distribution. If we can show that this limit solves

$$\mathcal{L}u = F'(0)u \diamond \xi, \quad u(0) = u_0 \tag{55}$$

for some white noise ξ , we can argue by uniqueness to finish the proof. We have

$$\mathcal{L}^\varepsilon \mathcal{E}^\varepsilon u^\varepsilon = \mathcal{E}^\varepsilon(F^\varepsilon(u^\varepsilon)(\xi^\varepsilon - c^\varepsilon F'(0))),$$

where we already know, by considering the same decomposition as in Lemma 5.7, that the right hand side is bounded in $\mathcal{M}_T^\alpha \mathcal{C}_p^{2\alpha+2\kappa/\sigma-2}(\mathbb{R}^d, e_l^\sigma)$ and converges due to the (\mathcal{E}) property of the objects on the right hand side in distribution in a weaker space to $F'(0)u \diamond \xi$. The convergence of the left hand side follows from Lemma 3.4. \square

Since the weights we are working with are increasing, the solutions u^ε and the limit u are actually classical tempered distributions. However, since we need the \mathcal{S}_ω spaces to handle convolutions in e_l^σ weighted spaces it is natural to allow for solutions in \mathcal{S}'_ω . An exception is the case where ξ^ε is Gaussian, since then it can be handled by a logarithmic weight (compare [AC15, Lemma 5.3]) and therefore e_l^σ could be replaced by a time-dependent polynomial weight. In the linear case, $F = \text{Id}$, we can allow for sub-exponentially growing initial conditions u_0 since the only reason for choosing the parameter l in the weight e_{l+t}^σ smaller than $-T$ was to be able to estimate $e_{l+t}^\sigma \leq (e_{l+t}^\sigma)^2$ to handle the quadratic term. In this case the solution will be a genuine ultra-distribution.

A Appendix

Results related to Section 2

Proof of Lemma 2.10. It is straightforward to check $f \cdot g \in C_\omega^\infty(U)$ for $f, g \in C_\omega^\infty(U)$ using Leibniz's rule. For the stability under composition see e.g. [RS12, Proposition 3.1], from which the stability under division can be easily derived.

For the identity (19) see [Bjö66, Example 1.5.7], the proof goes essentially as in (A.1) below.

The stability of $\mathcal{S}_\omega(\mathbb{R}^d)$ under addition, multiplication and convolution are quite easy to check. [Bjö66, Proposition 1.8.3].

For the inclusion $\mathcal{S}_\omega(\mathbb{R}^d) \subseteq C_\omega^\infty(\mathbb{R}^d)$ take for $f \in \mathcal{S}_\omega(\mathbb{R}^d)$ an arbitrary compact set K a larger, compact set $K' \supset \supset K$ and a test function $\chi \in \mathcal{D}_\omega(K')$ s.t. $\chi|_K = 1$ (for its existence see Lemma 2.12 below).

We then have by stability of multiplication in $\mathcal{S}_\omega(\mathbb{R}^d)$ that $\chi f \in \mathcal{D}_\omega(\mathbb{R}^d)$ and can then apply (19). \square

Lemma A.1. *The mappings $(\mathcal{F}, \mathcal{F}^{-1})$ as defined in subsection 2.3 map the spaces $(\mathcal{S}_\omega(\mathcal{G}), \mathcal{S}_\omega(\hat{\mathcal{G}}))$ and $(\mathcal{S}'_\omega(\mathcal{G}), \mathcal{S}'_\omega(\hat{\mathcal{G}}))$ to each other.*

Proof. We only consider the non-standard case $\omega = |\cdot|^\sigma$. Given $f \in \mathcal{S}_\omega(\hat{\mathcal{G}})$ the sequence

$$\hat{f}(x) = |\mathcal{G}| \sum_{k \in \mathcal{G}} f(k) e^{2\pi i k x}$$

does obviously converge to a smooth function that is periodic on $\hat{\mathcal{G}}$. We estimate on $\hat{\mathcal{G}}$ (and thus on every compact set)

$$\left| \partial^\alpha \sum_{k \in \mathcal{G}} |\mathcal{G}| f(k) e^{2\pi i k x} \right| \lesssim_\lambda \sum_{k \in \mathcal{G}} |\mathcal{G}| |k|^{|\alpha|} e^{-\lambda |k|^\sigma}$$

We can use Lemma A.3 for $|\cdot|^{|\alpha|} e^{-\lambda |\cdot|^{1/s}}$ with $\Omega = \mathcal{G}$ and $c > 0$ of the form $c = C(\lambda) \cdot C^{|\alpha|}$ (C denoting a positive constant that may change from line to line) which yields

$$\left| \partial^\alpha \sum_{k \in \mathcal{G}} |\mathcal{G}| f(k) e^{2\pi i k x} \right| \lesssim_\lambda C^{|\alpha|} \int_{\mathbb{R}^d} |x|^{|\alpha|} e^{-\lambda |x|^\sigma} dx$$

We now proceed as in [Hör05, Lemma 12.7.4] and estimate the integral by the Γ -function

$$\begin{aligned} \int_{\mathbb{R}^d} |x|^{|\alpha|} e^{-\lambda |x|^\sigma} dx &\lesssim \int_0^\infty r^{|\alpha|+d-1} e^{-\lambda r^\sigma} dr \lesssim_\lambda \lambda^{-s|\alpha|} \int_0^\infty r^{|\alpha|+d-1} e^{-r^\sigma} dr \\ &\lesssim \lambda^{-|\alpha|/\sigma} \Gamma((|\alpha|+d-1)/\sigma) \stackrel{\text{Stirling}}{\lesssim} \lambda^{-|\alpha|/\sigma} C^{|\alpha|} |\alpha|^{|\alpha|/\sigma}. \end{aligned}$$

Since we can choose $\lambda > 0$ arbitrarily large we see that indeed $f \in C_\omega^\infty(\hat{\mathcal{G}})$.

For the opposite direction, $f \in \mathcal{S}_\omega(\hat{\mathcal{G}})$, we use that by integration by parts for $z \in \mathcal{G}$, $l \geq 0$, $i = 1, \dots, d$ $\left| z_i^l \cdot \check{f}(z) \right| \lesssim C^l \sup_{\hat{\mathcal{G}}} (\partial^i)^l f \lesssim C^l \varepsilon^l l^{1/\sigma}$ With Stirling's formula and Lemma 3.7 we then obtain $\left| \check{f}(z) \right| \lesssim e^{\lambda |z|^\sigma}$. This shows the statement for the pair $(\mathcal{S}_\omega(\mathcal{G}), \mathcal{S}_\omega(\hat{\mathcal{G}}))$. The estimates above show that $\mathcal{F}, \mathcal{F}^{-1}$ are in fact continuous w.r.t to the corresponding topologies so that the statement for the dual spaces $(\mathcal{S}'_\omega(\mathcal{G}), \mathcal{S}'_\omega(\hat{\mathcal{G}}))$ immediately follows. \square

Lemma A.2. *Let \mathcal{G}^ε be as in definition 2.2. In each of the cases $j = -1, j \in \{0, \dots, j_{\mathcal{G}^\varepsilon} - 1\}$, $j = j_{\mathcal{G}^\varepsilon}$ there exists a $\mathcal{K} \in \mathcal{S}_\omega(\mathbb{R}^d)$, independent of ε , such that*

$$K_j(x) = \mathcal{F}^{-1} \varphi_j^{\mathcal{G}^\varepsilon}(x) = 2^{jd} \cdot \mathcal{K}(2^j x), \quad x \in \mathcal{G}^\varepsilon.$$

Proof. For $j < j_{\mathcal{G}^\varepsilon}$ we can simply take $2^{jd} \mathcal{K}(2^j \cdot) := \mathcal{F}_{\mathbb{R}^d}^{-1} \varphi_j^{\mathcal{G}^\varepsilon} = \mathcal{F}_{\mathbb{R}^d}^{-1} \varphi_j$, where φ_j denotes the partition of unity from which $\varphi_j^{\mathcal{G}^\varepsilon}$ was constructed, compare page 13. For the case $j = j_{\mathcal{G}^\varepsilon}$ we can choose a $k \in \mathbb{Z}$, independent of $\varepsilon = 2^{-N}$, such that $2^N = 2^k \cdot 2^{j_{\mathcal{G}^\varepsilon}}$. We can therefore write $\varphi_{j_{\mathcal{G}^\varepsilon}}^{\mathcal{G}^\varepsilon} = \sum_{j < j_{\mathcal{G}^\varepsilon}} \varphi_j = \Phi(2^{-N} \cdot)$ with a $\Phi \in \mathcal{S}_\omega(\widehat{\mathcal{G}^\varepsilon})$. Using the smear function ψ from subsection 2.5 we thus obtain

$$K_{j_{\mathcal{G}^\varepsilon}}^{\mathcal{G}^\varepsilon} = \mathcal{F}_{\mathbb{R}^d}^{-1} (\psi(2^{-N} \cdot) \Phi(2^{-N} \cdot)) = 2^{Nd} \mathcal{F}_{\mathbb{R}^d}^{-1} \overbrace{(\psi \Phi)}^{\in \mathcal{D}_\omega(\mathbb{R}^d)} (2^N \cdot) = 2^{Nd} \mathcal{K}(2^N \cdot).$$

with $\mathcal{K} = \mathcal{F}_{\mathbb{R}^d}^{-1}(\psi \Phi)$. □

Lemma A.3. *Given a lattice \mathcal{G} as in (2) we denote the translations of the closed parallelotope $G := [0, 1]a_1 + \dots + [0, 1]a_d$ by $\mathbb{G} := \{g + G \mid g \in \mathcal{G}\}$. Let $\Omega \subseteq \mathcal{G}$ and set $\bar{\Omega} := \bigcup_{G' \in \mathbb{G}, G' \cap \Omega \neq \emptyset} G'$. If for a measurable function $f : \bar{\Omega} \rightarrow \mathbb{R}_+$ there is a $c \geq 1$ such that for any $g \in \Omega$ there is a $G'(g) \in \mathbb{G}$, $g \in G'(g)$ with $f(g) \leq c \cdot \text{ess inf}_{x \in G'} f(x)$ then it also holds*

$$\sum_{g \in \Omega} |\mathcal{G}| f(g) \leq c \cdot 2^d \int_{\bar{\Omega}} f(x) dx.$$

Proof. Indeed

$$\begin{aligned} \sum_{g \in \Omega} |\mathcal{G}| f(g) &\leq c \sum_{g \in \Omega} \int_{G'(g)} f(x) dx \leq c \sum_{g \in \Omega} \sum_{G' \in \mathbb{G}, g \in G'} \int_{G'(g)} f(x) dx \\ &= c \sum_{G' \in \mathbb{G}, G' \subseteq \bar{\Omega}} \sum_{g \in \Omega, g \in G'} \int_{G'} f(x) dx \stackrel{(\Delta)}{=} 2^d c \sum_{G' \in \mathbb{G}} \int_{G'} f(x) dx = 2^d c \int_{\bar{\Omega}} f(x) dx, \end{aligned}$$

where we used in (Δ) that the d -dimensional parallelotope has 2^d vertices. □

Lemma A.4. *We have for $j \in \mathbb{N}_{>0}$ and $\alpha_1, \dots, \alpha_j \in \mathbb{N}_{>0}$*

$$j! \alpha_1! \dots \alpha_j! \leq (\alpha_1 + \dots + \alpha_j)!$$

Proof. This follows from a simple combinatorial argument: Let $k = \alpha_1 + \dots + \alpha_j$. Then while the right hand side corresponds to the number of arbitrary orderings of k elements, the left hand side corresponds to the number of possibilities to arrange these elements while keeping them together in sets of size $\alpha_1, \dots, \alpha_j$. □

Lemma A.5 (Mixed Young inequality). *For $f : \mathbb{R}^d \rightarrow \mathbb{C}$ and $g : \mathcal{G} \rightarrow \mathbb{C}$ we set for $x \in \mathbb{R}^d$*

$$f * g(x) := \sum_{k \in \mathcal{G}} |\mathcal{G}| f(x - k) g(k)$$

Then for $r, p, q \in [1, \infty]$ with $1 + 1/r = 1/p + 1/q$

$$\|f * g\|_{L^r(\mathbb{R}^d)} \leq \sup_{x \in \mathbb{R}^d} \|f(x - \cdot)\|_{L^p(\mathcal{G})}^{1 - \frac{p}{r}} \cdot \|f\|_{L^p(\mathbb{R}^d)}^{\frac{p}{r}} \|g\|_{L^q(\mathcal{G})}.$$

(with the convention $1/\infty = 0, \infty/\infty = 1$).

Proof. We assume $p, q, r \in (1, \infty)$. The remaining cases are easy to check.

The proof is based on Hölder's inequality on \mathcal{G} with $\frac{1}{r} + \frac{1}{\frac{rp}{r-p}} + \frac{1}{\frac{rq}{r-q}} = 1$

$$\begin{aligned} |f * g(x)| &\leq \sum_{k \in \mathcal{G}} |\mathcal{G}| (|f(x-k)|^p |g(k)|^q)^{1/r} \cdot |f(x-k)|^{\frac{r-p}{r}} |g(k)|^{\frac{r-q}{r}} \\ &\stackrel{\text{Hölder}}{\leq} \left\| (|f(x-\cdot)|^p |g(\cdot)|^q)^{1/r} \right\|_{L^r(\mathcal{G})} \cdot \| |f(x-\cdot)|^{\frac{r-p}{r}} \|_{L^{\frac{rp}{r-p}}(\mathcal{G})} \cdot \| |g(\cdot)|^{\frac{r-q}{r}} \|_{L^{\frac{rq}{r-q}}(\mathcal{G})} \\ &= \left(\sum_{k \in \mathcal{G}} |\mathcal{G}| (|f(x-k)|^p |g(k)|^q) \right)^{1/r} \sup_{x' \in \mathbb{R}^d} \|f(x'-\cdot)\|_{L^p(\mathcal{G})}^{\frac{r-p}{r}} \|g\|_{L^q(\mathcal{G})}^{\frac{r-q}{r}} \end{aligned}$$

Raising this expression to the r th power and integrating it shows the claim. \square

Results related to Section 3

Lemma A.6. *For $t \geq 0$, $p \in [1, \infty]$, $\rho \in \boldsymbol{\rho}(\omega)$ we have on compact time intervals*

$$\|e^{tL^\varepsilon} \varphi\|_{L^p(\mathcal{G}^\varepsilon, \rho)} \lesssim \|\varphi\|_{L^p(\mathcal{G}^\varepsilon, \rho)}.$$

and for $\beta > 0$

$$\|e^{tL^\varepsilon} \varphi\|_{L^p(\mathcal{G}^\varepsilon, \rho)} \lesssim t^{-\beta/2} \|\varphi\|_{C_p^{-\beta}(\mathcal{G}^\varepsilon, \rho)}$$

uniformly in ε .

Proof. With the random walk $(X_t)_{t \in \mathbb{R}_+}$ which is generated by L^ε on \mathcal{G} we can express the semigroup as $e^{tL^\varepsilon} f(x) = \mathbb{E}[f(x + \varepsilon X_{t/\varepsilon^2})]$ so that by Jensen's inequality

$$\begin{aligned} \sum_{x \in \mathcal{G}^\varepsilon} |\mathcal{G}^\varepsilon| |\rho(x) e^{tL^\varepsilon} f(x)|^p &\leq \mathbb{E} \left[\sum_{x \in \mathcal{G}^\varepsilon} |\mathcal{G}^\varepsilon| |\rho(x) f(x + \varepsilon X_{t/\varepsilon^2})|^p \right] \\ &\stackrel{(20)}{\lesssim} \mathbb{E} \left[e^{p\omega(\varepsilon X_{t/\varepsilon^2})} \sum_{x \in \mathcal{G}^\varepsilon} |\mathcal{G}^\varepsilon| |f(x + \varepsilon X_{t/\varepsilon^2}) \rho(x + \varepsilon X_{t/\varepsilon^2})|^p \right] = \mathbb{E} \left[e^{p\omega(\varepsilon X_{t/\varepsilon^2})} \right] \|f\|_{L^p(\mathcal{G}^\varepsilon, \rho)}^p. \end{aligned}$$

Application of the next lemma finishes the proof of the first estimate. The second estimate follows as in Lemma 6.6. of [GP15b]. \square

Lemma A.7. *The random walk generated by L^ε on \mathcal{G}^ε satisfies for any $c, c' > 0$ and $t \in [0, T]$*

$$\mathbb{E}[e^{c\omega(|X_t^\varepsilon|)}] \lesssim_{c, c'} e^{c'\omega(t)}.$$

Proof. We assume $\omega = |\cdot|^\sigma$, if ω is of the polynomial form (14) the proof follows by similar, but simpler arguments. We write shorthand $s = 1/\sigma$.

By the Lévy-Khintchine-formula we have for $\theta \in \mathbb{R}$ $\mathbb{E}[e^{i\theta X_t^\varepsilon}] = e^{-t/\varepsilon^2 \int_{\mathcal{G}} (1 - e^{i\theta \varepsilon x}) d\mu(x)} = e^{-t l^\varepsilon(\theta)}$. We want to bound first for $k \geq 1$

$$\mathbb{E}[|X_{t,1}^\varepsilon|^k + \dots + |X_{t,d}^\varepsilon|^k] = \sum_{i=1}^d \left| \partial_{\theta_i}^k |_{\theta=0} \mathbb{E}[e^{i\theta X_t^\varepsilon}] \right|$$

To this end we apply Faà-di-Brunos formula with $u(v) = e^{-tv}$, $v(\theta) = l^\varepsilon(\theta)$. Note that with Lemma 3.5

$$\begin{aligned} u^{(m)}(1) &= (-t)^m \\ |\partial_{\theta_i}^{\alpha_i} v(0)| &\lesssim_\delta \delta^{|\alpha_i|} (\alpha_i!)^s \end{aligned}$$

Thus with $A_{m,k} = \{\alpha \in \mathbb{N}_{>0}^m \mid \sum_i \alpha_i = k\}$ for some $\delta \in (0, 1]$

$$\begin{aligned} \left| \partial_{\theta_i}^k |_{\theta=0} \mathbb{E}[e^{i\theta X_t^\varepsilon}] \right| &= \left| \sum_{1 \leq m \leq k, \alpha \in A_{m,k}} \frac{k!}{m! \alpha!} u^{(m)}(1) \prod_{i=1}^m \partial_{\theta_i}^{\alpha_i} v(0) \right| \\ &\lesssim \sum_{1 \leq m \leq k, \alpha \in A_{m,k}} \frac{k!}{m! \alpha!} t^m \prod_{i=1}^m (\alpha_i!)^s \delta^{|\alpha_i|} \leq \delta^k \sum_{1 \leq m \leq k, \alpha \in A_{m,k}} t^m k! (m!)^{s-1} \prod_{i=1}^m (\alpha_i!)^{s-1} \\ &\stackrel{\text{Lemma A.4}}{\leq} \delta^k (k!)^s \sum_{1 \leq m \leq k, \alpha \in A_{m,k}} t^m = \delta^k (k!)^s \sum_{1 \leq m \leq k} \binom{k-1}{m-1} t^m \\ &= \delta^k (k!)^s t (1+t)^{k-1} \leq \delta^k (k!)^s (1+t)^k \end{aligned}$$

With $|x|_k^k := |x_1|^k + \dots + |x_d|^k$ we get

$$\mathbb{E}[|X_t^\varepsilon|_k^k] \lesssim \delta^k (k!)^s (1+t)^k$$

and therefore, using Stirling's formula and $|x|^k \lesssim C^k \cdot |x|_k^k$ (with a generic constant $C > 0$ as usual),

$$\begin{aligned} \mathbb{E}[e^{c|X_t^\varepsilon|^\sigma}] &\lesssim 1 + \mathbb{E}[e^{c|X_t^\varepsilon|^\sigma} \mathbf{1}_{|X_t^\varepsilon| \geq 1}] \leq 1 + \sum_{n=0}^{\infty} \frac{c^n}{n!} \mathbb{E}[|X_t^\varepsilon|^{[n\sigma]}] \lesssim 1 + \sum_{n=0}^{\infty} \frac{C^n t^{[n\sigma]}}{n^n} \delta^{[n\sigma]} [n\sigma]^{[n\sigma]s} \\ &\lesssim 1 + t \sum_{n=0}^{\infty} \frac{C^n \delta^{n\sigma} t^{n\sigma}}{n^n} n^n = 1 + t e^{C \delta^\sigma t^\sigma} \end{aligned}$$

Choosing $\delta > 0$ small enough finishes the proof. \square

Lemma A.8. *The object*

$$\nabla \phi \ll \nabla \psi(t, x) := \frac{1}{2} \sum_{0 \leq i \leq j \leq \varepsilon} \int_{\mathbb{R}^d} \frac{d\mu(y)}{\varepsilon^2} [Q_i S_{i-1} \phi(t, x + \varepsilon y) - Q_i S_{i-1} \phi(t, x)] \cdot [\Delta_i \psi(t, x + \varepsilon y) - \Delta_i \psi(t, x)]$$

satisfies the bound

$$t^\gamma \|\nabla \phi \ll \nabla \psi\|_{C_p^{\alpha+\beta-2}(\mathcal{G}^\varepsilon, \rho_1, \rho_2)} \lesssim \|\phi\|_{\mathcal{M}_t^\gamma C_p^\alpha(\mathcal{G}^\varepsilon, \rho_1)} \|\psi\|_{C_\infty^\beta(\mathcal{G}^\varepsilon, \rho_2)}$$

for $\alpha \in (0, 1)$, $\beta \in \mathbb{R}$, $p \in [1, \infty]$, $\gamma \in [0, 1)$ and $s \mapsto \rho_1(s), \rho_2(s)$ pointwise decreasing.

Proof. We can reshape $\nabla \phi \ll \nabla \psi$ as $\int_{\mathbb{R}^d} \frac{d\mu(y)}{\varepsilon^2} \bar{\phi}^y \ll \bar{\psi}^y$ where $\bar{\phi}^y(t, x) = \phi(t, x + \varepsilon y) - \phi(t, x)$ and similar for ψ . The bound therefore follow from Lemma 4.5 once we can show

$$\|\bar{\phi}^y\|_{C_p^{\alpha-1}(\mathcal{G}^\varepsilon, \rho_1)} \lesssim \|\phi\|_{C_p^\alpha(\mathcal{G}^\varepsilon, \rho_1)} |y| \varepsilon \quad (56)$$

for any $\alpha \in \mathbb{R}$. Note that due to Lemma A.2 we can write

$$\Delta_j \phi = (K_j(\cdot + \varepsilon y) - K_j) * \phi.$$

where $K_j = 2^{jd} \mathcal{K}(2^j \cdot)$ with $\mathcal{K} \in \mathcal{S}_\omega(\mathbb{R}^d)$ depending on the case $j \in \{-1\}, \{0, \dots, j_{\mathcal{G}^\varepsilon}\}$ and $j_{\mathcal{G}^\varepsilon}$. With

$$K_j(x + \varepsilon y) - K_j(x) = 2^{-j} \int_0^1 2^{jd} \mathcal{K}(2^j(x + t\varepsilon y)) \cdot y \varepsilon$$

we get (56) by applying Lemma 2.16. \square

Results related to Section 5

Proof of Lemma 5.4. This is a consequence of the results in [CSZ17]. For $z \in \mathcal{G}^\varepsilon$ let $G^\varepsilon(z) = z + [-\varepsilon/2, \varepsilon/2)a_1 + \dots + [-\varepsilon/2, \varepsilon/2)a_d$, where a_1, \dots, a_d denote the vectors that span \mathcal{G} . For $x \in \mathbb{R}^d$ let $[x] := z$ be the (unique) element $z \in \mathcal{G}^\varepsilon$ such that $x \in G^\varepsilon(z)$ and for $x \in (\mathbb{R}^d)^k$ set $[x] = ([x_1], \dots, [x_k])$. We will start by showing

$$\lim_{\varepsilon \rightarrow 0} \|f_k^\varepsilon([\cdot]) - f_k\|_{L^2((\mathbb{R}^d)^k)} = 0 \quad (57)$$

for all k .

By Parseval's identity we have $\|f_k^\varepsilon([\cdot]) - f_k\|_{L^2((\mathbb{R}^d)^k)} = \|\mathcal{F}_{(\mathbb{R}^d)^k}(f_k^\varepsilon([\cdot])) - \hat{f}_k\|_{L^2((\mathbb{R}^d)^k)}$, where $\mathcal{F}_{(\mathbb{R}^d)^k}$ denotes the Fourier transform on $(\mathbb{R}^d)^k$ for which we get

$$\mathcal{F}_{(\mathbb{R}^d)^k}(f_k^\varepsilon([\cdot])) = \widehat{f_k^\varepsilon p_k^\varepsilon},$$

where we recall that $\widehat{f_k^\varepsilon}$ is the discrete Fourier transform of f_k^ε which we interpret as usual as a periodic function (on $(\mathbb{R}^d)^k$) and where

$$p_k^\varepsilon(y_1, \dots, y_k) = \int_{G^1(0)^k} \frac{dz_1 \dots dz_k}{|\mathcal{G}^1|^k} e^{2\pi i \varepsilon (y_1 \cdot z_1 + \dots + y_k \cdot z_k)}.$$

The function p_k^ε is uniformly bounded and tends to 1 as ε goes to 0. Now we apply once Parseval's identity on $(\mathbb{R}^d)^k$ and once on $(\widehat{\mathcal{G}^\varepsilon})^k$ and obtain

$$\begin{aligned} \int_{(\mathbb{R}^d)^k} dx_1 \dots dx_k \left| (\widehat{f_k^\varepsilon p^\varepsilon})(x_1, \dots, x_k) \right|^2 &= \sum_{z_1, \dots, z_k \in \mathcal{G}^\varepsilon} |\mathcal{G}^\varepsilon|^k |f_k^\varepsilon(z_1, \dots, z_k)|^2 \\ &= \int_{(\widehat{\mathcal{G}^\varepsilon})^k} dx_1 \dots dx_k \left| \widehat{f_k^\varepsilon}(x_1, \dots, x_k) \right|^2 \end{aligned}$$

and thus

$$\int_{((\widehat{\mathcal{G}^\varepsilon})^k)^c} dx_1 \dots dx_k \left| (\widehat{f_k^\varepsilon p^\varepsilon})(x_1, \dots, x_k) \right|^2 = \int_{(\widehat{\mathcal{G}^\varepsilon})^k} dx_1 \dots dx_k (|\widehat{f_k^\varepsilon}|^2 (1 - |p^\varepsilon|^2)(x_1, \dots, x_k)).$$

Since $\mathbf{1}_{(\widehat{\mathcal{G}^\varepsilon})^k} \widehat{f_k^\varepsilon}$ is uniformly in ε bounded by the $L^2((\mathbb{R}^d)^k)$ function g_k and since $1 - |p^\varepsilon|^2$ converges pointwise to zero, it follows from the dominated convergence theorem that $\mathbf{1}_{((\widehat{\mathcal{G}^\varepsilon})^k)^c} \widehat{f_k^\varepsilon} p^\varepsilon$ converges to zero in $L^2((\mathbb{R}^d)^k)$. Thus, we get

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \|\widehat{f_k^\varepsilon} p_k^\varepsilon - \hat{f}_k\|_{L^2((\mathbb{R}^d)^k)} &= \lim_{\varepsilon \rightarrow 0} \|\mathbf{1}_{(\widehat{\mathcal{G}^\varepsilon})^k} \widehat{f_k^\varepsilon} p_k^\varepsilon - \hat{f}_k\|_{L^2((\mathbb{R}^d)^k)} \\ &\leq \lim_{\varepsilon \rightarrow 0} \|(\mathbf{1}_{(\widehat{\mathcal{G}^\varepsilon})^k} \widehat{f_k^\varepsilon} - \hat{f}_k) p_k^\varepsilon\|_{L^2((\mathbb{R}^d)^k)} + \lim_{\varepsilon \rightarrow 0} \|\hat{f}_k (1 - p_k^\varepsilon)\|_{L^2((\mathbb{R}^d)^k)} = 0, \end{aligned}$$

where for the first term we used that p_k^ε is uniformly bounded in ε and that by assumption $\mathbf{1}_{(\widehat{\mathcal{G}}^\varepsilon)^k} \widehat{f}_k^\varepsilon$ converges to \widehat{f}_k in $L^2((\mathbb{R}^d)^k)$ and for the second term we combined the fact that p_k^ε converges pointwise to 1 with the dominated convergence theorem. We have therefore shown (57). Note that this implies

$$\|f_k^\varepsilon([\cdot]) \mathbf{1}_{\forall i \neq j \ z_i \neq z_j} - f_k\|_{L^2(\mathbb{R}^d)} \rightarrow 0 \quad \& \quad \|f_k^\varepsilon([\cdot]) \mathbf{1}_{\exists i \neq j \ z_i = z_j}\|_{L^2(\mathbb{R}^d)} \rightarrow 0. \quad (58)$$

As in the proof of Lemma 5.1 we identify \mathcal{G}^ε with some arbitrary enumeration $\mathbb{Z} \rightarrow \mathcal{G}^\varepsilon$ and use the set $A_r^k = \{a \in \mathbb{N}_0^r \mid \sum_i a_i = k\}$ so that we can write

$$\mathcal{J}_k f_k^\varepsilon = \sum_{1 \leq r \leq k, a \in A_r^k} \binom{k}{a} \sum_{z_1 < \dots < z_r} |\mathcal{G}^\varepsilon|^k \tilde{f}_{\varepsilon,a}^k(z_1, \dots, z_r) \cdot \prod_{j=1}^r :\xi(z_j)^{a_j}:$$

where we denote as in the proof of Lemma 5.1 by $\tilde{f}_{\varepsilon,a}^k$ the symmetrized restriction of f_ε^k to $(\mathbb{R}^d)^r$. By Theorem 2.3 of [CSZ17] we see that the $r = k$ term of $\mathcal{J}_k f_k^\varepsilon$ converges due to 58 to the desired limit in distribution, so that we only have to show that the remaining terms vanish as ε tends to 0. The idea is to redefine the noise in these terms by $\bar{\xi}_j^\varepsilon(z) = :\xi(z)^{a_j}:/r_j^\varepsilon(z)$ where $r_j^\varepsilon(z) := \sqrt{\text{Var}(:\xi(z)^{a_j}:) \cdot |\mathcal{G}^\varepsilon|} \lesssim |\mathcal{G}^\varepsilon|^{(1-a_j)/2}$, so that in view of [CSZ17, Lemma 2.3] it suffices to show that

$$\sum_{z_1 < \dots < z_r} |\mathcal{G}^\varepsilon|^r \prod_{j=1}^r r_j^\varepsilon(z_j)^2 \cdot |\tilde{f}_{\varepsilon,a}^k(z_1, \dots, z_r)|^2 \lesssim \sum_{z_1 < \dots < z_r} |\mathcal{G}^\varepsilon|^k \cdot |\tilde{f}_{\varepsilon,a}^k(z_1, \dots, z_r)|^2 \rightarrow 0,$$

but this follows from (58). \square

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